
On Loop Algebras

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Preamble

This essay develops the structure and representation theory of a large class of infinite-dimensional Lie algebras, commonly referred to as loop algebras. These have become central objects in the general representation theory of infinite-dimensional Lie algebras and have found a number of applications in both pure mathematics and theoretical physics.

The exposition is divided four-fold. First, section 1 constructs the class of loop algebras and develops the necessary structure results. Thereafter, sections 2 and 3 develop their representation theory. Lastly, section 4 generalises the construction of section 1 to a larger class of Lie algebras. The reader is assumed to be familiar with the classical representation theory of finite-dimensional Lie algebras, the homology theory of Lie algebras, and the usual formative curriculum in mathematics.

Throughout this essay, the ground field is taken to be $\mathbb{C}$. We, furthermore, follow the standard conventions in representation theory of taking modules to be left modules, ideals to be left ideals, *et cetera*. For a Lie algebra $\mathfrak{g}$, we denote by $\mathcal{U}(\mathfrak{g})$ its universal enveloping algebra. Lastly, any reference to an infinite cardinality is understood to mean a countable infinite, which we denote by $\aleph_0$. The remaining notation follows that of the standard mathematical convention.

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Bibliographical Notes

This essay lays no claim to originality. Rather, the presented results are collected from the wide and fruitful literature that has established itself in the theory of loop algebras over the last decades. In particular, section 1 draws primarily on [Kac 1994] with contributions by [Allison, Berman & Pianzola 2004]. Thereafter, section 2 uses results from [Chari & Pressley 2001], [Chari & Moura 2004], [Senesi 2009], [Amini 2012], and [Rao 1993]. Some of the general category-theoretic results therein are taken from [Etingof & Moura 2003]. In the sequel, section 3 rests on [Chari, Fourier & Senesi 2008] and [Senesi 2010]. Lastly, section 4 draws primarily on [Kac, Raina & Rozhkovskaya 2013]. The cohomological results therein are taken from [Verdier 1982].

1 Construction and Structure

In the following, $\mathfrak{g}$ is taken to be a semisimple, finite-dimensional Lie algebra. We go through with the general construction of loop algebras.

Def. 1.1. Define the set $\text{aut}(\mathfrak{g}) \subset \text{Aut}(\mathfrak{g})$ of Lie algebra automorphisms of finite order. Let $\psi \in \text{aut}(\mathfrak{g})$. We say that $n \in \mathbb{Z}_{>0}$ is a period of ψ if $\psi^n \equiv \text{id}$. We denote the set of periods of ψ by $\text{Per}(\psi)$.

Let $\psi \in \text{aut}(\mathfrak{g})$ and $n \in \text{Per}(\psi)$. Let ξ be a primitive n^{th} root of unity. Then $\mathfrak{g}$ decomposes into ψ -eigenspaces with eigenvalues $\xi^{\mathbb{Z}_n}$

$$\mathfrak{g} = \bigoplus_{j=0}^{n-1} \mathfrak{g}_j, \quad \text{where } \mathfrak{g}_j = \{x \in \mathfrak{g} \mid \psi(x) = \xi^j x\}. \quad (1.1)$$

Denote by $[\cdot]_{\text{mod } n} : \mathbb{Z} \rightarrow \mathbb{Z}_n$ the reduction modulo n . We can now naturally define the following Lie algebra.

Def. 1.2. Define the n^{th} twisted loop algebra with respect to ψ

$$\mathfrak{L}(\mathfrak{g}, \psi, n) = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}_{[j]_{\text{mod } n}} \otimes t^j \quad (1.2)$$

with Lie bracket for $x, y \in \mathfrak{g}$ and $i, j \in \mathbb{Z}$ given by

$$[x \otimes t^i, y \otimes t^j] = [x, y] \otimes t^{i+j},$$

where on the right the Lie bracket is the one on $\mathfrak{g}$.

Technically, this construction depends on our numbering of the roots of unity. However, by fixing such an ordering once and for all, we may ignore potential ambiguities in the above definition. We now define a special case of this general construction: the untwisted loop algebra of $\mathfrak{g}$, which we will generally refer to simply as the loop algebra of $\mathfrak{g}$. Denote by $\mathfrak{L} = \mathbb{C}[t, t^{-1}]$ the algebra of Laurent polynomials over $\mathbb{C}$. Define further $\mathfrak{L}^n = \mathbb{C}[t^n, t^{-n}] \subset \mathfrak{L}$.

Def. 1.3. Define the untwisted loop algebra $\mathfrak{L}(\mathfrak{g}) := \mathfrak{L}(\mathfrak{g}, \text{id}, 1) \cong \mathfrak{g} \otimes_{\mathbb{C}} \mathfrak{L}$.

Lem. 1.4. Let $\psi \in \text{aut}(\mathfrak{g}); n \in \text{Per}(\psi)$. Then $\mathfrak{L}(\mathfrak{g}, \psi, n) \leq \mathfrak{L}(\mathfrak{g})$ is a Lie subalgebra.

Proof. We may extend ψ to an automorphism $\tilde{\psi}$ of $\mathfrak{L}(\mathfrak{g})$ by the assignment

$$\tilde{\psi} : \mathfrak{L}(\mathfrak{g}) \rightarrow \mathfrak{L}(\mathfrak{g}); x \otimes t^k \mapsto \xi^{-k} \psi(x) \otimes t^k,$$

extending linearly. Since we may equivalently describe this map as $\tilde{\psi} = \psi \otimes \xi$, wherein $\xi(t) = \xi^{-1}t$, this is clearly well-defined. For $x \in \mathfrak{g}_{[k] \bmod n}$ we have

$$\tilde{\psi}(x \otimes t^k) = \xi^{-k} \psi(x) \otimes t^k = \xi^{-k} \xi^{[k] \bmod n} x \otimes t^k = x \otimes t^k,$$

by definition of the k -eigenspace. We see that we may define the twisted loop algebra $\mathfrak{L}(\mathfrak{g}, \psi, n)$ as

$$\mathfrak{L}(\mathfrak{g}, \psi, n) = \{a \in \mathfrak{L}(\mathfrak{g}) \mid \tilde{\psi}(a) = a\},$$

the set of fixed points of $\mathfrak{L}(\mathfrak{g})$ under the $\tilde{\psi}$ -action. Since the set of fixed points under a Lie algebra automorphism is itself a Lie subalgebra, the claim follows. $\square$

There exists the following equivalence.

Lem. 1.5. *Denote by $|\psi|$ the order of ψ . Then $\mathfrak{L}(\mathfrak{g}, \psi, n) \cong \mathfrak{L}(\mathfrak{g}, \psi, |\psi|)$.*

Proof. Note that $|\psi|$ is a divisor of n . If we denote by $\xi_{|\psi|}, \xi_n$ the respective primitive roots of unity in the eigenspace decompositions, we may rewrite $\xi_{|\psi|} = \xi_n^{\frac{n}{|\psi|}}$. We construct an explicit isomorphism. Consider the map

$$\Psi : \mathfrak{L}(\mathfrak{g}, \psi, |\psi|) \rightarrow \mathfrak{L}(\mathfrak{g}, \psi, n); x \otimes t^j \mapsto x \otimes t^{\frac{n}{|\psi|}j},$$

extending $\mathbb{C}$ -linearly. Clearly, Ψ is an injective $\mathbb{C}$ -algebra homomorphism. Furthermore

$$\Psi(t^m(x \otimes t^j)) = \Psi(x \otimes t^{m+j}) = x \otimes t^{\frac{n}{|\psi|}m + \frac{n}{|\psi|}j} = t^{\frac{n}{|\psi|}m} \Psi(x \otimes t^j) = \Psi(t^m) \Psi(x \otimes t^j),$$

proving that Ψ is $\mathbb{C}[t, t^{-1}]$ -linear. Since the sets of eigenvalues of ψ in the decompositions $\mathfrak{L}(\mathfrak{g}, \psi, n)$ and $\mathfrak{L}(\mathfrak{g}, \psi, |\psi|)$ are identical, it follows that Ψ is surjective, and hence a $\mathbb{C}$ -algebra isomorphism. To conclude that Ψ is a Lie algebra homomorphism, we note that the Lie bracket is given by the commutator. $\square$

Cor. 1.6. *Let $m, n \in \text{Per}(\psi)$. Then $\mathfrak{L}(\mathfrak{g}, \psi, m) \cong \mathfrak{L}(\mathfrak{g}, \psi, n)$.*

We may thus ease the notation as $\mathfrak{L}(\mathfrak{g}, \psi) := \mathfrak{L}(\mathfrak{g}, \psi, n)$ without danger of ambiguity. The following clarifies why we may call $\mathfrak{L}(\mathfrak{g})$ the loop algebra.

Cor. 1.7. *Let $n \in \mathbb{Z}_{>0}$. Then $\mathfrak{L}(\mathfrak{g}) \cong \mathfrak{L}(\mathfrak{g}, \text{id}, n)$.*

Lem. 1.8. *Consider the loop algebra $\mathfrak{L}(\mathfrak{g}, \psi)$. Then $\mathfrak{g}_0$ in the decomposition from equations (1.1) and (1.2) is a reductive Lie subalgebra of $\mathfrak{g}$.*

Proof. Let $\mathfrak{h}' \subset \mathfrak{g}_0$ be a maximal $\text{ad}(\mathfrak{g})$ -diagonalisable subalgebra. Let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra containing $\mathfrak{h}'$. Consider the root decomposition $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$ with respect to this Cartan subalgebra $\mathfrak{h}$ and set $\Delta' \subset \Delta$ as the root subspace such that $\forall \alpha \in \Delta' : \alpha|_{\mathfrak{h}'} = 0$. The centraliser $C_{\mathfrak{g}}(\mathfrak{h}')$ of $\mathfrak{h}'$ in $\mathfrak{g}$ is, then, of the form

$$C_{\mathfrak{g}}(\mathfrak{h}') = \mathfrak{h} + \bigoplus_{\alpha \in \Delta'} \mathfrak{g}_\alpha = \mathfrak{h} \oplus \mathfrak{s},$$

where $\mathfrak{s}$ is a semisimple Lie algebra whose intersection with $\mathfrak{g}_0$ is trivial. Since $\mathfrak{h}'$ is abelian, $\mathfrak{h}' \subset C_{\mathfrak{g}}(\mathfrak{h}')$, whence $\mathfrak{h}' \subset \mathfrak{h}$. Therefore, the $\mathfrak{h}$ -decomposition restricts onto $\mathfrak{g}_0$ and we find that $\mathfrak{g}_0$ is reductive. $\square$

Lem. 1.9. *Let $\mathfrak{h}' \subset \mathfrak{g}_0$ be a maximal $\text{ad}(\mathfrak{g})$ -diagonalisable subalgebra as in lem. 1.8. Then the centraliser $C_{\mathfrak{g}}(\mathfrak{h}')$ of $\mathfrak{h}'$ in $\mathfrak{g}$ is a Cartan subalgebra of $\mathfrak{g}$.*

Proof. By the proof of lem. 1.8, the centraliser of $\mathfrak{h}'$ decomposes as

$$C_{\mathfrak{g}}(\mathfrak{h}') = \mathfrak{h} \oplus \mathfrak{s},$$

where $\mathfrak{h} \leq \mathfrak{g}$ is the Cartan subalgebra containing $\mathfrak{h}'$ and $\mathfrak{s}$ is a semisimple Lie algebra whose intersection with $\mathfrak{g}_0$ is trivial. In fact, by the construction in the proof of lem. 1.8, $\mathfrak{s}$ is a ψ -invariant subalgebra. Therefore, the $\mathbb{Z}_n$ -grading of equation (1.1) induces a $\mathbb{Z}_n$ -grading of $\mathfrak{s}$ with the property that $\mathfrak{s}$ is the trivial space. We show by induction on i that for all $i \in \mathbb{Z}_n : \mathfrak{s}_i = 0$. The claim holds by the above for $i = 0$. Let $i > 0$ and take $x \in \mathfrak{s}_{[i]_{\text{mod } n}}$. Then, $(\text{ad}(x))^r \mathfrak{s}_{[j]_{\text{mod } n}} \subset \mathfrak{s}_{[ir+j]_{\text{mod } n}}$. Choose r such that $i(r-1) < n-j \leq ir$. Then, $ir+j = n+t$ where $0 \leq t < i$. By the inductive assumption, $\mathfrak{s}_{[ir+j]_{\text{mod } n}} = \mathfrak{s}_{[t]_{\text{mod } n}} = 0$, whence $\text{ad}(x)$ is nilpotent. By the same token, $\text{ad}(y)$ for $y \in \mathfrak{s}_{[-i]_{\text{mod } n}}$ is nilpotent. Since $[\mathfrak{s}_{[i]_{\text{mod } n}}, \mathfrak{s}_{[-i]_{\text{mod } n}}] \subset \mathfrak{s}_0 = 0$, $\text{ad}(x)$ and $\text{ad}(y)$ commute. Furthermore, the Cartan-Killing form K gives, by the nilpotency of the adjoint action, $K(\text{ad}(x), \text{ad}(y)) = 0$. By the Cartan criterion, K is non-degenerate due to the semisimplicity of $\mathfrak{s}$. Therefore, it must hold that $\mathfrak{s}_n = 0$. We conclude the claim by induction, whence $C_{\mathfrak{g}}(\mathfrak{h}') = \mathfrak{h}$. $\square$

Recall that to a simple, finite-dimensional Lie algebra $\mathfrak{g}$ we may associate a Cartan matrix $A \in \text{Mat}(n, \mathbb{C})$, which carries the information of the root system of $\mathfrak{g}$. Associate to this A the index set of $\mathfrak{g}$ -vertices $I_A = \{1, \dots, n\}$. A bijection $\tau : I_A \rightarrow I_A$ on this index set is called a diagram automorphism of I_A if for all $i, j \in \{1, \dots, n\}$ we have $A_{ij} = A_{\tau(i)\tau(j)}$. Recall further that we may equip $\mathfrak{g}$ with a Chevalley basis $\{e_i, h_i, f_i \mid i = 1, \dots, n\}$ such that each of the subalgebras spanned by respective pairs e_i, h_i, f_i is isomorphic to $\mathfrak{sl}_2(\mathbb{C})$. To this Chevalley basis and a given diagram automorphism τ of I_A as above, we associate the unique Lie algebra automorphism $\tau \in \text{aut}(\mathfrak{g})$, defined by

$$\tau(e_i) = e_{\tau(i)} \quad \tau(f_i) = f_{\tau(i)}. \quad (1.3)$$

Def. 1.10. Let $\mu \in \text{aut}(\mathfrak{g})$. We call μ a diagram automorphism of $\mathfrak{g}$ if it is conjugate to an automorphism of the form as in equation (1.3).

More simply, we may think of diagram automorphisms as automorphisms of the root system that preserve the structure and orientation of the Dynkin diagram associated with $\mathfrak{g}$. Unraveling the definitions, this is equivalent to the explicit construction in def. 1.10. Going through the Dynkin diagrams of complex, finite-dimensional, simple Lie algebras and checking all possible finite-order automorphisms that preserve the diagram, we find that the only types that admit a non-trivial diagram automorphism are A_n , D_n , and E_6 . All types except for D_4 admit a unique non-trivial diagram automorphism of order 2, whereas D_4 admits one non-trivial automorphism of order 2 and one of order 3. These considerations are not used presently, but will be useful in section 3 for the representation theory. The following results clarify the usefulness of considering diagram automorphisms.

Prop. 1.11. Let $\mathfrak{g}$ be a simple, finite-dimensional Lie algebra with Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$, as constructed in lem. 1.9 from a maximal $\text{ad}(\mathfrak{g})$ -diagonalisable subalgebra $\mathfrak{h}'$, and simple roots $\Pi = \{\alpha_1, \dots, \alpha_n\}$. Let $\psi \in \text{aut}(\mathfrak{g})$ with $m \in \text{Per}(\psi)$. Then ψ is conjugate to

$$\mu \cdot \exp(\text{ad}(\frac{2\pi i}{m}h)) \in \text{aut}(\mathfrak{g})$$

for a diagram automorphism $\mu \in \text{aut}(\mathfrak{g})$ that preserves $\mathfrak{h}$. Here, $h \in \mathfrak{h}^\mu$ lies in the fixed point set of μ in $\mathfrak{h}$, and $\forall i \in \{1, \dots, n\} : \langle \alpha_i, h \rangle \in \mathbb{Z}$.

Proof. By lem. 1.9, a maximal $\text{ad}(\mathfrak{g})$ -diagonalisable subalgebra $\mathfrak{h}' \leq \mathfrak{g}_0$ must contain a regular element $x \in \mathfrak{g}$ since we are working over the ground field $\mathbb{C}$, over which every Cartan subalgebra is generated by the generalised eigenspace $\text{Hau}(\text{ad}(x), 0)$ to the zero eigenvalue of the adjoint action $\text{ad}(x)$ of a regular element x . Then, the centraliser $C_{\mathfrak{g}}(x) = \mathfrak{h}$ is a ψ -invariant Cartan subalgebra. Furthermore, the space

$$\begin{aligned} \mathfrak{n}_+ = \{y \in \mathfrak{g} \mid \exists a \in \mathbb{C} \text{ with } (\Re(a) > 0) \text{ or } (\Re(a) = 0 \wedge \Im(a) > 0) \\ \text{such that } \text{ad}(x)y = ay\} \end{aligned}$$

is a maximal, nilpotent, ψ -invariant $\mathfrak{g}$ -subalgebra. Indeed, this corresponds simply to the positive weight space in a triangular decomposition of a finite-dimensional Lie algebra. Denote by Δ_+ the set of positive roots in this decomposition. Since $\mathfrak{n}_+$ is nilpotent and ψ -invariant, ψ induces a finite-order automorphism of Δ_+ . This, in turn, induces an automorphism μ of the associated Dynkin diagram through the ψ -induced action on the simple roots Π . Then

$$\psi \cdot \mu^{-1} = \exp(\frac{2\pi i}{m}\text{ad}(h))$$

since $\psi \cdot \mu^{-1}$ is a finite-order, fixed-point automorphism of the simple roots with eigenvalues corresponding to those of the ψ -action on $\mathfrak{g}$. These are precisely given by the collection $\exp(\frac{2\pi i}{m} \text{ad}(h))$ for $h \in \mathfrak{h}'$ since the diagonal action of $\mathfrak{h}' \subset \mathfrak{h}$ on the simple roots Π is induced by the one of $\mathfrak{h}$ and is, hence, integer. $\square$

Lem. 1.12. *Let $\psi, \phi \in \text{aut}(\mathfrak{g})$ with same period $m \in \mathbb{N}$. Let $\gamma : \mathbb{C}^\times \rightarrow \text{Aut}(\mathfrak{L}(\mathfrak{g}))$ be a path. We view γ as an element of $\text{Aut}(\mathfrak{L}(\mathfrak{g}))$ by the point-wise action of $\gamma(t) \in \text{Aut}(\mathfrak{L}(\mathfrak{g}))$. Let $\xi = \exp(\frac{2\pi i}{m})$ be a primitive m^{th} root of unity. Then*

$$\gamma(t)\mathfrak{L}(\mathfrak{g}, \psi) = \mathfrak{L}(\mathfrak{g}, \phi)$$

if and only if

$$\forall t \in \mathbb{C}^\times : \phi \gamma(\xi^{-1}t) = \gamma(t)\psi,$$

where we view any element in $\text{aut}(\mathfrak{g})$ as an element of $\text{Aut}(\mathfrak{L}(\mathfrak{g}))$ by extending it $\mathfrak{L}$ -linearly.

Proof. This follows directly from lem. 1.4 by unraveling the alternative definition of $\mathfrak{L}(\mathfrak{g}, \psi)$ as a subalgebra of $\mathfrak{L}(\mathfrak{g})$. $\square$

In particular, lem. 1.12 gives a classification of the isomorphism classes of twisted loop subalgebras $\mathfrak{L}(\mathfrak{g}, \psi) \leq \mathfrak{L}(\mathfrak{g})$.

Prop. 1.13. *Let $\psi \in \text{aut}(\mathfrak{g})$ be a finite-order automorphism. Then $\mathfrak{L}(\mathfrak{g}, \psi) \cong \mathfrak{L}(\mathfrak{g}, \mu)$ for some diagram automorphism $\mu \in \text{aut}(\mathfrak{g})$.*

Proof. By prop. 1.11, ψ is conjugate to a finite-order automorphism of the form

$$\mu \cdot \exp(\text{ad}(\frac{2\pi i}{m}h)),$$

where μ is a diagram automorphism. Consider the path $t^h : \mathbb{C}^\times \rightarrow \text{Aut}(\mathfrak{g})$ that acts on $\mathfrak{g}_\alpha$ in the root space decomposition of $\mathfrak{g}$ as multiplication by $t^{\langle \alpha, h \rangle}$ and extend $\mathfrak{L}$ -linearly to $\text{Aut}(\mathfrak{L}(\mathfrak{g}))$. Then

$$t^h \mu = \mu t^h,$$

whence, by lem. 1.12, we conclude

$$t^h \mathfrak{L}(\mathfrak{g}, \psi) = \mathfrak{L}(\mathfrak{g}, \mu).$$

By construction, the Lie structure is preserved under this isomorphism. We conclude the claim. $\square$

Thus, we may henceforth assume that any twisted loop algebra $\mathfrak{L}(\mathfrak{g}, \psi)$ is defined via a diagram automorphism ψ . Furthermore, the isomorphism class of $\mathfrak{L}(\mathfrak{g}, \psi)$ depends only on the connected component of $\text{aut}(\mathfrak{g})$ containing ψ . The following is, then, an immediate corollary of lem. 1.8.

Cor. 1.14. *Let $\psi \in \text{aut}(\mathfrak{g})$ be a diagram automorphism. Then, the subalgebra $\mathfrak{g}_0 \leq \mathfrak{g}$ in the decomposition of equation (1.1) is simple.*

In particular, we use this to give a classification of ideals in $\mathfrak{L}(\mathfrak{g}, \psi)$.

Lem. 1.15. *Let $\psi \in \text{aut}(\mathfrak{g})$ be a diagram automorphism of order r . Suppose $\mathfrak{i} \trianglelefteq \mathfrak{L}(\mathfrak{g}, \psi)$ is an ideal. Then $\mathfrak{i}$ is of the form $\mathfrak{i} = P\mathfrak{L}(\mathfrak{g}, \psi)$ for some $P \in \mathfrak{L}^r$. In particular, maximal ideals are of the form $(1 - (at)^r)\mathfrak{L}(\mathfrak{g}, \psi)$ for some $a \in \mathbb{C}^\times$, and the codimension of any non-zero ideal $\mathfrak{i} \trianglelefteq \mathfrak{L}(\mathfrak{g}, \psi)$ is finite.*

Proof. Let $\mathfrak{i} \trianglelefteq \mathfrak{L}(\mathfrak{g}, \psi)$ be a non-trivial ideal and let $\mathfrak{h}_0 \subset \mathfrak{g}_0$ be a Cartan subalgebra, whose existence is guaranteed by cor. 1.14. A general element $x \in \mathfrak{i}$ is of the form

$$x = \sum_{i \in I} \sum_{j \in J} a_{i,j} \otimes t^{[j] \bmod r} P_{i,j},$$

where $I, J \subset \mathbb{Z}$ are index sets, $a_{i,j} \in \mathfrak{g}_{[j] \bmod r}$ are linearly independent and $P_{i,j} \in \mathfrak{L}$ are non-zero. We claim that for any $Q \in \mathfrak{L}^r$, we have

$$QP_{i,j}\mathfrak{L}(\mathfrak{g}, \psi) \subset \mathfrak{i}.$$

We may assume that x is an eigenvector of $\text{ad}(\mathfrak{h}_0)$ to some weight $\alpha \in \mathfrak{h}_0^*$; otherwise, we restrict x iteratively onto those parts for which it is an eigenvector of the $\mathfrak{h}_0$ adjoint action. Suppose $\alpha \neq 0$. We may, then, reduce the problem to $\alpha = 0$ by replacing x with

$$[x, a_{-j} \otimes t^j],$$

where $a_{-j} \in \mathfrak{g}_{[-j] \bmod r}$. Thus, $j = 0$ and all $a_{i,j} \in \mathfrak{h}_0$. Take a weight $\beta \in \mathfrak{h}_0^*$ such that there exists some $a_{i,j}$ with $\langle \beta, a_{i,j} \rangle \neq 0$, denote by $e_{\pm\beta}$ weight vectors with weights $\pm\beta$, and set $y \in \mathfrak{i}$ to be

$$y := [[x, e_\beta \otimes Q], e_{-\beta}].$$

By the above, y admits an expansion

$$y = h_0 \otimes P_{i,j} + \sum_{k=1}^{r-1} h_k \otimes t^k P_k$$

where $h_0 \in \mathfrak{h}_0$ is non-zero, $h_k \in \mathfrak{g}_{[k] \bmod r}$ have zero weight with respect to the $\mathfrak{h}_0$ -adjoint action, $P_k \in \mathfrak{L}$. From this we see that for any weight vector $e_l \in \mathfrak{g}_0$, it holds that $[y, e_l] \in \mathfrak{i}$. Therefore, $QP_{i,j} \otimes \mathfrak{h}_0 \subset \mathfrak{i}$, whence we conclude the claim. Let $z \in \mathfrak{g}_{[j] \bmod r}$. It follows that

$$\mathfrak{i}_0 = \{p \in \mathfrak{L} \mid z \otimes t^j p \in \mathfrak{i}\} \trianglelefteq \mathfrak{L}$$

is an invariant ideal that does not depend on j and z . Combining with the above, we find that $\mathfrak{i} = \mathfrak{i}_0 \mathfrak{L}(\mathfrak{g}, \psi)$. We show that $\mathfrak{L}^r$ is a principal ideal domain. Recall that the polynomial ring $\mathbb{C}[t]$ is a principal ideal domain. Letting $S = \{t^{r^j} \mid j \in \mathbb{N}\}$, we may write $\mathfrak{L}^r = S^{-1} \mathbb{C}[t]$ as the localisation of the polynomial ring. Since the localisation of a principal ideal domain is, again, a principal ideal domain, the claim follows. From principality of $\mathfrak{L}^r$, we conclude that $\mathfrak{i} = P \mathfrak{L}$ for some $P \in \mathfrak{L}^r$. The claim for maximal ideals, then, follows from the Nullstellensatz for $\mathfrak{L}^r$. $\square$

Cor. 1.16. *Let $\mathfrak{i} \trianglelefteq \mathfrak{L}(\mathfrak{g}, \psi)$ be a non-zero ideal. Then $\mathfrak{L}(\mathfrak{g}, \psi)/\mathfrak{i}$ is a finite-dimensional Lie algebra.*

Cor. 1.17. *Let $\mathfrak{i} \trianglelefteq \mathfrak{L}(\mathfrak{g})$ be an ideal. Then $\mathfrak{i} = \mathfrak{g} \otimes_{\mathbb{C}} \mathfrak{j}$ for some ideal $\mathfrak{j} \trianglelefteq \mathfrak{L}$.*

Proof. Recall $\mathfrak{L}(\mathfrak{g}) = \mathfrak{g} \otimes_{\mathbb{C}} \mathfrak{L}$. The statement follows from the fact that $\mathfrak{L}$ is a principal ideal domain and from lem. 1.15, since the order of the diagram automorphism $\text{id} \in \text{aut}(\mathfrak{g})$ is 1, whence the given $P \in \mathfrak{L}$ must be the sole generator of $\mathfrak{j}$. $\square$

Recall that $\mathfrak{g}$ admits the triangular decomposition $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$, where $\mathfrak{h} \subset \mathfrak{g}$ is the Cartan subalgebra of $\mathfrak{g}$.

Lem. 1.18. *Let $\psi \in \text{aut}(\mathfrak{g})$ and suppose that $\mathfrak{n}_+, \mathfrak{h}, \mathfrak{n}_-$ are invariant under ψ . Define the triangular decomposition*

$$\mathfrak{L}(\mathfrak{g}, \psi) = \mathfrak{L}(\mathfrak{n}_+, \psi) \oplus \mathfrak{L}(\mathfrak{h}, \psi) \oplus \mathfrak{L}(\mathfrak{n}_-, \psi).$$

This decomposition is well-defined.

Proof. Let n be the order of ψ . Expanding definitions on the right of equation (1.2), we find by distributivity of the tensor product and commutativity of the direct sum

$$\begin{aligned} & \bigoplus_{j \in \mathbb{Z}} \mathfrak{n}_+ [j]_{\text{mod } n} \otimes t^j \oplus \bigoplus_{k \in \mathbb{Z}} \mathfrak{h} [k]_{\text{mod } n} \otimes t^k \oplus \bigoplus_{l \in \mathbb{Z}} \mathfrak{n}_- [l]_{\text{mod } n} \otimes t^l \\ &= \bigoplus_{j \in \mathbb{Z}} \left(\mathfrak{n}_+ [j]_{\text{mod } n} \oplus \mathfrak{h} [j]_{\text{mod } n} \oplus \mathfrak{n}_- [j]_{\text{mod } n} \right) \otimes t^j \\ &= \bigoplus_{j \in \mathbb{Z}} \mathfrak{g} [j]_{\text{mod } n} \otimes t^j, \end{aligned}$$

since eigenspaces restrict onto invariant subspaces. The claim follows. $\square$

Cor. 1.19. *There is a triangular decomposition $\mathfrak{L}(\mathfrak{g}) = \mathfrak{L}(\mathfrak{n}_+) \oplus \mathfrak{L}(\mathfrak{h}) \oplus \mathfrak{L}(\mathfrak{n}_-)$.*

Cor. 1.20. *If $\psi \in \text{aut}(\mathfrak{g})$ is a diagram automorphism, $\mathfrak{L}(\mathfrak{g}, \psi)$ admits a triangular decomposition of the form from lem. 1.18.*

We turn to the study of 1-dimensional central extensions of $\mathfrak{L}(\mathfrak{g})$, though we will not pursue this construction much further than some heuristics on the structure. Note that a general element of $\mathfrak{L}(\mathfrak{g})$ has the form $a(t) = \sum_{k \in I} a_k \otimes t^k$, where $I \subset \mathbb{Z}$ is a finite index set and $a_k \in \mathfrak{g}$ for all k .

Thm. 1.21. $H_{alg}^2(\mathfrak{L}(\mathfrak{g}); \mathbb{C})$, is 1-dimensional and is spanned by the 2-cocycle ζ , defined for $a(t), b(t) \in \mathfrak{L}(\mathfrak{g})$ by

$$\zeta(a(t), b(t)) = \text{Res}_{t=0} \langle da(t), b(t) \rangle = \text{Res}_{t=0} \langle a'(t), b(t) \rangle,$$

where d denotes the exterior derivation.

Proof. [Weibel 1994, 7.9.6]. □

Using this result, we may, therefore, define a non-trivial 1-dimensional central extension of $\mathfrak{L}(\mathfrak{g})$ where the Lie bracket is induced by the 2-cocycle ζ as above.

Def. 1.22. Define the current algebra $\tilde{\mathfrak{L}}(\mathfrak{g})$ as the 1-dimensional central extension of the loop algebra

$$\tilde{\mathfrak{L}}(\mathfrak{g}) = \mathfrak{L}(\mathfrak{g}) \oplus \mathbb{C}K,$$

where K is central and the bracket is given, for $a(t), b(t) \in \mathfrak{L}(\mathfrak{g}), \mu, \lambda \in \mathbb{C}$, by

$$[a(t) + \lambda K, b(t) + \mu K] = [a(t), b(t)] + \text{Res}_{t=0} \langle a'(t), b(t) \rangle K,$$

where the bracket to the right is the one on $\mathfrak{g}$.

By thm. 1.21, this is well-defined up to isomorphism.

Rem. 1.23. We briefly explain the motivation for calling $\tilde{\mathfrak{L}}(\mathfrak{g})$ a current algebra. The terminology originates from the physics literature. Current algebras were introduced by M. Gell-Mann (1964) as a tool to describe the force currents in fundamental particle interactions. Using the notation $x^{(n)} = x \otimes t^n$ and introducing the formal parameter z , we may associate to $x \in \mathfrak{g}$ a formal generating series

$$x(z) = \sum_{n \in \mathbb{Z}} x^{(n)} z^{-n-1},$$

called the current, or quantum field, of x .

We could now try to define an invariant symmetric bilinear form on $\tilde{\mathfrak{L}}(\mathfrak{g})$, but we would quickly run into trouble with degeneracy of this form. For, since $\mathfrak{g}$ is semisimple, it is perfect and thus satisfies $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$. We, then, have that $[\mathfrak{L}(\mathfrak{g}), \mathfrak{L}(\mathfrak{g})] = \mathfrak{L}(\mathfrak{g})$. As such, any element in $\mathfrak{L}(\mathfrak{g})$ can be expressed as a sum of terms $[x, y] \otimes t^k$ for $x, y \in \mathfrak{g}$ and $k \in \mathbb{Z}$. Suppose, then, that $\tilde{\mathfrak{L}}(\mathfrak{g})$ admits a bilinear form $\langle \cdot, \cdot \rangle$ that is invariant under its adjoint action. Then $\langle [x, y], K \rangle = \langle x, [y, K] \rangle = 0$, since K is central in $\tilde{\mathfrak{L}}(\mathfrak{g})$. As $y \in \mathfrak{g}$ is arbitrary, this form is degenerate in K . For this reason, it is useful to enlarge $\tilde{\mathfrak{L}}(\mathfrak{g})$ by an exterior derivation, allowing us to define a non-degenerate form on the resulting structure.

Def. 1.24. Define the affinisation $\hat{\mathfrak{L}}(\mathfrak{g})$ of $\mathfrak{L}(\mathfrak{g})$ by the vector space extension

$$\hat{\mathfrak{L}}(\mathfrak{g}) = \tilde{\mathfrak{L}}(\mathfrak{g}) \oplus \mathbb{C}d = \mathfrak{L}(\mathfrak{g}) \oplus \mathbb{C}K \oplus \mathbb{C}d,$$

where d denotes the exterior derivation that acts on $\mathfrak{L}(\mathfrak{g})$ as $t \frac{d}{dt}$ and kills K . The Lie bracket on $\hat{\mathfrak{L}}(\mathfrak{g})$ is given, for $x, y \in \mathfrak{g}$ and $\mu_1, \mu_2, \lambda_1, \lambda_2 \in \mathbb{C}$, by

$$\begin{aligned} [x \otimes t^m \oplus \lambda_1 K \oplus \mu_1 d, y \otimes t^n \oplus \lambda_2 K \oplus \mu_2 d] = \\ = ([x, y] \otimes t^{m+n} + y \otimes \mu_1 n t^n - x \otimes \mu_2 m t^m) \oplus m \delta_{m,-n} \langle x, y \rangle K, \end{aligned}$$

where the bilinear form to the right is the one on $\mathfrak{g}$.

More simply, one may define the affinisation $\hat{\mathfrak{L}}(\mathfrak{g}) = \tilde{\mathfrak{L}}(\mathfrak{g}) \rtimes \mathbb{C}d$ as the semidirect product. Setting the degrees of K and d to zero, $\hat{\mathfrak{L}}(\mathfrak{g})$ naturally inherits a $\mathbb{Z}$ -grading from equation (1.2).

Rem. 1.25. The reason for naming $\hat{\mathfrak{L}}(\mathfrak{g})$ an affinisation is due to it being an explicit construction of the untwisted, affine Kac-Moody algebras, as introduced by V. Kac and R. Moody in 1967 by a construction generalising that of E. Cartan and W. Killing for finite-dimensional Lie algebras. The Kac-Moody algebras of affine type are precisely the class of infinite-dimensional Lie algebras with finite Gelfand-Kirillov dimension obtained from this construction. Therefore, they form a particularly well-behaved class of infinite-dimensional Lie algebras and carry representations with wide-ranging, important applications.

Given our symmetric invariant bilinear form on $\mathfrak{g}$, we may canonically extend it to $\mathfrak{L}(\mathfrak{g})$ via $\langle a \otimes t^m, b \otimes t^n \rangle = \langle a, b \rangle \otimes t^{m+n}$. We use this to define a symmetric invariant bilinear form on $\hat{\mathfrak{L}}(\mathfrak{g})$.

Prop. 1.26. $\hat{\mathfrak{L}}(\mathfrak{g})$ carries a non-degenerate symmetric invariant bilinear form $\langle \cdot, \cdot \rangle$ defined by

$$\begin{aligned} \langle a(t), b(t) \rangle &= \text{Res}_{t=0}(t^{-1} \langle a(t), b(t) \rangle_{\mathfrak{L}(\mathfrak{g})}), \\ \langle K, a(t) \rangle &= 0, \quad \langle K, K \rangle = 0, \\ \langle d, a(t) \rangle &= 0, \quad \langle d, K \rangle = 1, \quad \langle d, d \rangle = 0, \end{aligned}$$

for $a(t), b(t) \in \mathfrak{L}(\mathfrak{g})$, and where we extended the bilinear form from $\mathfrak{g}$ to $\mathfrak{L}(\mathfrak{g})$.

Proof. It is clear that this form is non-degenerate, as guaranteed by the condition $\langle d, K \rangle = 1$. Invariance can be checked simply by direct calculation. $\square$

The analogous construction also goes through in the twisted case, wherein all of the results still hold. These twisted affinisations are of great import as they are isomorphic to the affine Kac-Moody algebras of twisted type, and thus complete their geometric construction. However, we will not pursue this in greater detail.

2 Representation Theory of Untwisted Loop Algebras $\mathfrak{L}(\mathfrak{g})$

This section addresses the representation theory of general $\mathfrak{L}(\mathfrak{g})$. Note that by the canonical inclusion

$$\mathfrak{g} \hookrightarrow \mathfrak{L}(\mathfrak{g}); x \mapsto x \otimes 1$$

any $\mathfrak{L}(\mathfrak{g})$ -module restricts to a representation of $\mathfrak{g}$. It will turn out that the irreducible, finite-dimensional representations of $\mathfrak{L}(\mathfrak{g})$ can be fully classified. Given a finite-dimensional $\mathfrak{g}$ -module V , we construct representations of $\mathfrak{L}(\mathfrak{g})$.

Def. 2.1. Let V be a finite-dimensional $\mathfrak{g}$ -module. Let $z \in \mathbb{C}^\times$. Define the evaluation representation $V(z)$ of $\mathfrak{L}(\mathfrak{g})$

$$\text{ev} : \mathfrak{L}(\mathfrak{g}) \rightarrow \text{End}(V(z)); x \otimes t^m \mapsto z^m x,$$

where the action of a is the one of the $\mathfrak{g}$ -representation.

Since $\mathfrak{L}(\mathfrak{g})$ is a Lie algebra, this representation acts via derivation. Thus, we may also state the explicit form on $\bigotimes_{i=1}^n V_i(z_i)$ for $z_i \in \mathbb{C}^\times$ and $\mathfrak{g}$ -modules V_i

$$\text{ev} : \mathfrak{L}(\mathfrak{g}) \rightarrow \text{End}\left(\bigotimes_{i=1}^n V_i(z_i)\right); x \otimes t^m \mapsto \sum_{i=1}^n \text{id}^{\otimes(i-1)} \otimes z_i^m x \otimes \text{id}^{\otimes(n-i)}.$$

These tensor representations are the key to classifying irreducible $\mathfrak{L}(\mathfrak{g})$ -modules.

Prop. 2.2. *The evaluation representation $\bigotimes_{i=1}^n V_i(z_i)$ is irreducible as an $\mathfrak{L}(\mathfrak{g})$ -module if and only if all $z_i \in \mathbb{C}^\times$ are distinct.*

Proof. Suppose first that all $z_i \in \mathbb{C}^\times$ are distinct. The problem then reduces to the irreducibility of the representation

$$\varrho : \mathfrak{g}^{\oplus n} \rightarrow \text{End}\left(\bigotimes_{i=1}^n V_i\right); (x_1, \dots, x_n) \mapsto x_1 \otimes \dots \otimes x_n$$

since we may factor out by the $z_i \in \mathbb{C}^\times$. However, $\bigotimes_{i=1}^n V_i$ as a product of irreducible $\mathfrak{g}$ -modules V_i is irreducible as a $\mathfrak{g}^{\oplus n}$ -module due to $\mathfrak{g}$ being semisimple.

We show the converse direction via contraposition. First recall from the theory of finite-dimensional Lie algebras that for irreducible, non-trivial representations V of $\mathfrak{g}$, the product $V \otimes V^*$ contains $\mathfrak{g} \oplus \mathbb{C}$. Let V_1, V_2 be two such representations of $\mathfrak{g}$. But then, since $\dim_{\mathbb{C}}(\mathfrak{g}) \geq 1$, we have

$$\dim_{\mathbb{C}}(\text{End}(V_1 \otimes V_2)) = \dim_{\mathbb{C}}((V_1 \otimes V_2) \otimes (V_1 \otimes V_2)^*)$$

$$\begin{aligned}
&= \dim_{\mathbb{C}}((V_1 \otimes V_1^*) \otimes (V_2 \otimes V_2^*)) \\
&\geq 2.
\end{aligned}$$

However, for V an irreducible $\mathfrak{g}$ -module, it must hold that $\dim_{\mathbb{C}}(\text{End}(V)) = 1$. Therefore, if $z_1 = z_2$, $V_1(z_1) \otimes V_2(z_2)$ is reducible, since $V_1(z) \otimes V_2(z) \cong (V_1 \otimes V_2)(z)$ and $(V_1 \otimes V_2)$ is then reducible by the above. We conclude the claim by contraposition. $\square$

It turns out that the evaluation representations are not only a subset of the irreducible $\mathfrak{L}(\mathfrak{g})$ -modules, but that they exhaust them.

Thm. 2.3. *Any irreducible, finite-dimensional $\mathfrak{L}(\mathfrak{g})$ -module V is of the form*

$$V \cong \bigotimes_{i=1}^n V_i(z_i)$$

for irreducible, finite-dimensional $\mathfrak{g}$ -representations V_i with all $z_i \in \mathbb{C}^\times$ distinct.

Proof. Let $\varrho : \mathfrak{L}(\mathfrak{g}) \rightarrow \text{End}(V)$ be an irreducible representation. Since ϱ is a Lie algebra homomorphism, its kernel $\ker(\varrho) \trianglelefteq \mathfrak{L}(\mathfrak{g})$ is an ideal. By lem. 1.15 and cor. 1.17, $\text{codim}(\ker(\varrho)) < \infty$ and $\ker(\varrho) = \mathfrak{g} \otimes_{\mathbb{C}} \mathfrak{j}$ for $\mathfrak{j} \trianglelefteq \mathfrak{L}$ an ideal. By cor. 1.16, $\mathfrak{L}(\mathfrak{g})/\ker(\varrho)$ is a finite-dimensional Lie algebra. By construction, ϱ factors to a faithful representation of $\mathfrak{L}(\mathfrak{g})/\ker(\varrho)$. Thus, since $\mathfrak{L}(\mathfrak{g})/\ker(\varrho)$ admits a faithful, irreducible, finite-dimensional representation, it is reductive. Due to semisimplicity of $\mathfrak{g}$, we have $[\mathfrak{L}(\mathfrak{g}), \mathfrak{L}(\mathfrak{g})] = \mathfrak{L}(\mathfrak{g})$. In turn, $[\mathfrak{L}(\mathfrak{g})/\ker(\varrho), \mathfrak{L}(\mathfrak{g})/\ker(\varrho)] = \mathfrak{L}(\mathfrak{g})/\ker(\varrho)$. Therefore, $\mathfrak{L}(\mathfrak{g})/\ker(\varrho)$ is in fact semisimple.

We claim that $\mathfrak{j}$ is radical. To this end, let $P \in \sqrt{\mathfrak{j}}$ and let $k \in \mathbb{N}$ such that $P^k \in \mathfrak{j}$. Let $x \in \mathfrak{g}$ and consider the ideal $\mathfrak{k} = \langle x \otimes P \rangle$. The n^{th} term in the derived series of $\mathfrak{k}$, $\mathfrak{k}^{(n)}$, is itself an ideal, and, by construction of P , we have $\mathfrak{k}^{(n)} \subset \mathfrak{g} \otimes \mathfrak{j} = \ker(\varrho)$. Since $\mathfrak{k}$ is an ideal, $\mathfrak{k}^{(n)} \leq \mathfrak{k}$. Therefore $\mathfrak{k}^{(n)} \subset \mathfrak{k} \cap \ker(\varrho)$. By the third isomorphism theorem for Lie algebras, we find

$$(\mathfrak{k} + \ker(\varrho))/\ker(\varrho) \cong \mathfrak{k}/(\mathfrak{k} \cap \ker(\varrho)).$$

It follows that

$$((\mathfrak{k} + \ker(\varrho))/\ker(\varrho))^{(n)} \cong (\mathfrak{k}/(\mathfrak{k} \cap \ker(\varrho)))^{(n)} \subset (\mathfrak{k} \cap \ker(\varrho))/(\mathfrak{k} \cap \ker(\varrho)) \cong 0.$$

Thus, $(\mathfrak{k} + \ker(\varrho))/\ker(\varrho)$ is a solvable ideal of $\mathfrak{L}(\mathfrak{g})/\ker(\varrho)$ and is by definition contained in the radical $\text{rad}(\mathfrak{L}(\mathfrak{g})/\ker(\varrho))$. However, since $\mathfrak{L}(\mathfrak{g})/\ker(\varrho)$ is semisimple, $\text{rad}(\mathfrak{L}(\mathfrak{g})/\ker(\varrho)) = 0$, and thus $\mathfrak{k} \subset \ker(\varrho)$. In particular, $P \in \mathfrak{j}$ and $\sqrt{\mathfrak{j}} = \mathfrak{j}$.

Since $\mathfrak{L}$ is Noetherian, $\mathfrak{j}$ admits an irredundant primary decomposition $\mathfrak{j} = \bigcap_{i=1}^n Q_i$ for primary ideals $Q_i \trianglelefteq \mathfrak{L}$. But, since $\mathfrak{j}$ is radical, this irredundant primary decomposition $\mathfrak{j} = \bigcap_{i=1}^n Q_i$ descends to an irredundant prime decomposition. Since prime

ideals in $\mathfrak{L}$ are generated by linear polynomials of the form $(t - z)$ for $z \in \mathbb{C}^\times$, we find

$$\mathfrak{j} = \prod_{i=1}^n (t - z_i) \quad \text{for distinct } z_i \in \mathbb{C}^\times.$$

By the Chinese remainder theorem, we find for the quotient

$$\mathfrak{L} / \left(\prod_{i=1}^n (t - z_i) \right) \cong \bigoplus_{i=1}^n \mathfrak{L} / (t - z_i).$$

Furthermore, there is a clear isomorphism

$$\begin{aligned} \Psi : \mathfrak{g} \otimes_{\mathbb{C}} \bigoplus_{i=1}^n \mathfrak{L} / (t - z_i) &\rightarrow \mathfrak{g}^{\oplus n}; \\ (x_1 \otimes P_1(t), \dots, x_n \otimes P_n(t)) &\mapsto (P_1(z_1)x_1, \dots, P_n(z_n)x_n). \end{aligned}$$

Our representation ϱ factors through $\ker(\varrho)$ and we find

$$\mathfrak{L}(\mathfrak{g}) / \ker(\varrho) \cong (\mathfrak{g} \otimes_{\mathbb{C}} \mathfrak{L}) / (\mathfrak{g} \otimes_{\mathbb{C}} \mathfrak{j}) \cong \mathfrak{g} \otimes_{\mathbb{C}} (\mathfrak{L} / \mathfrak{j}) \cong \mathfrak{g} \otimes_{\mathbb{C}} \bigoplus_{i=1}^n \mathfrak{L} / (t - z_i) \cong \mathfrak{g}^{\oplus n}.$$

But, by a corollary of Schur's lemma, the irreducible representations of $\mathfrak{g}^{\oplus n}$ are precisely $\bigotimes_{i=1}^n V_i$ for V_i irreducible $\mathfrak{g}$ -modules. These $\mathfrak{L}(\mathfrak{g}) / \ker(\varrho)$ -modules are lifted to finite-dimensional irreducible representations of $\mathfrak{L}(\mathfrak{g})$ by requiring their restrictions to $\ker(\varrho) = \mathfrak{g} \otimes_{\mathbb{C}} \mathfrak{j}$ to vanish. These are precisely the representations of the form

$$\varrho : \mathfrak{L}(\mathfrak{g}) \rightarrow \text{End}\left(\bigotimes_{i=1}^n V_i(z_i)\right)$$

for distinct $z_i \in \mathbb{C}^\times$, as their irreducibility follows from prop. 2.2. $\square$

Finally, we give conditions on two evaluation representations being isomorphic.

Prop. 2.4. *Let $V_1, \dots, V_n, W_1, \dots, W_m$ be finite-dimensional $\mathfrak{g}$ -representations. Then there exists an isomorphism of $\mathfrak{L}(\mathfrak{g})$ -representations*

$$\bigotimes_{i=1}^n V_i(z_i) \cong \bigotimes_{j=1}^m W_j(z'_j)$$

if and only if $m = n$ and $W_i \cong V_i$, $z_i = z'_i$ up to pairwise permutation of indices.

Proof. Let $h \in \mathfrak{h} \subset \mathfrak{g}$ be an arbitrary element of the Cartan subalgebra. We want to consider the eigenvalues of $h \otimes t^k$ for all $k \in \mathbb{N}$. Since all V_i, W_j are finite-dimensional, they all admit vectors $v_i \in V_i, w_j \in W_j$ with highest h -eigenvalues

$\lambda_i(h), \mu_j(h)$ under the $\mathfrak{g}$ -action. Letting $h \otimes t^k$ act on the highest h -eigenvectors in the evaluation action, we find that for the isomorphism of representations to hold, we must have

$$\sum_{i=1}^n \lambda_i(h) z_i^k = \sum_{j=1}^m \mu_j(h) z_j'^k,$$

since Lie algebras act via derivation and the action of $\mathfrak{h}$ uniquely determines irreducible, finite-dimensional representations. Iterating over all $k \in \mathbb{N}$, introducing a formal parameter x , and taking these sums as coefficients of a power series in x^k , we find via geometric sum that

$$\sum_{i=1}^n \frac{\lambda_i(h)}{1 - x z_i} = \sum_{j=1}^m \frac{\mu_j(h)}{1 - x z_j'}$$

as functions of x . This gives us the desired conditions. $\square$

Together, thm. 2.3 and prop. 2.4 give a complete classification of irreducible, finite-dimensional $\mathfrak{L}(\mathfrak{g})$ -modules in terms of $\mathfrak{g}$ -modules. Furthermore, the classification of irreducible, finite-dimensional $\mathfrak{g}$ -modules is known from the classical representation theory. Thus, we have arrived at a full classification of irreducible, finite-dimensional $\mathfrak{L}(\mathfrak{g})$ -modules. Nevertheless, it is interesting to consider further, reducible representations of $\mathfrak{L}(\mathfrak{g})$.

Def. 2.5. Let $\mathbf{Rep}(\mathfrak{L}(\mathfrak{g}))$ be the category of finite-dimensional $\mathfrak{L}(\mathfrak{g})$ -modules, whose objects are finite-dimensional representations of $\mathfrak{L}(\mathfrak{g})$ and whose morphisms are equivariant maps.

It is clear that $\mathbf{Rep}(\mathfrak{L}(\mathfrak{g}))$ is an abelian tensor category. The following clarifies why it might be interesting to study reducible $\mathfrak{L}(\mathfrak{g})$ -modules.

Prop. 2.6. *The category $\mathbf{Rep}(\mathfrak{L}(\mathfrak{g}))$ is not semisimple.*

Proof. It is sufficient to find one $V \in \text{Obj}(\mathbf{Rep}(\mathfrak{L}(\mathfrak{g})))$ that is reducible but not decomposable. To this end, consider the quotient Lie algebra with canonical projection

$$\text{pr} : \mathfrak{L}(\mathfrak{g}) \rightarrow \mathfrak{g} \otimes \mathbb{C}[t]/(t^2).$$

Let W be a non-trivial, finite-dimensional $\mathfrak{g}$ -representation, whose existence is guaranteed by the semisimplicity of $\mathfrak{g}$. Consider the representation for $x, y \in \mathfrak{g}$

$$\varrho : \mathfrak{g} \otimes \mathbb{C}[t]/(t^2) \rightarrow \text{End}(W \oplus W); (x + yt)(w_1, w_2) = (xw_1, yw_1 + xw_2),$$

where $\mathfrak{g}$ -action on W . This representation is clearly reducible, since $0 \oplus W$ is a proper $\mathfrak{g} \otimes \mathbb{C}[t]/(t^2)$ -submodule. To show indecomposability, we claim that any

non-zero $\mathfrak{g} \otimes \mathbb{C}[t]/(t^2)$ -submodule W_0 of $W \oplus W$ contains $0 \oplus W$. Let $(w_1, w_2) \in W_0$. We may assume $w_1 \neq 0$, since otherwise there is nothing to show. Since W is an irreducible $\mathfrak{g}$ -representation, there is $y \in \mathfrak{g}$ such that $yw_1 \neq 0$. Then, $(yt)(w_1, w_2) = (0, yw_1) \in 0 \oplus W$ is non-zero. But, then, the claim follows, since $\mathfrak{U}(\mathfrak{g})w_1 = W$ due to irreducibility of W , whence, by the canonical inclusion

$$\mathfrak{g} \hookrightarrow \mathfrak{g} \otimes \mathbb{C}[t]/(t^2),$$

we have

$$0 \oplus W = (0, W) = \mathfrak{U}(\mathfrak{g})(0, w_1) \subset \mathfrak{U}(\mathfrak{g} \otimes \mathbb{C}[t]/(t^2))(0, w_1) \subset W_0.$$

It follows that $W \oplus W$ is indecomposable under the $\mathfrak{g} \otimes \mathbb{C}[t]/(t^2)$ -action. Therefore, $W \oplus W$ is a reducible, indecomposable $\mathfrak{g} \otimes \mathbb{C}[t]/(t^2)$ -module, which, in turn, lifts to a reducible, indecomposable $\mathfrak{L}(\mathfrak{g})$ -representation $V \in \text{Obj}(\mathbf{Rep}(\mathfrak{L}(\mathfrak{g})))$. $\square$

In particular, knowledge of the irreducible modules in $\mathbf{Rep}(\mathfrak{L}(\mathfrak{g}))$ is not sufficient to classify finite-dimensional $\mathfrak{L}(\mathfrak{g})$ -modules. We thus want to find a more general classification of objects in $\mathbf{Rep}(\mathfrak{L}(\mathfrak{g}))$. Though $\mathbf{Rep}(\mathfrak{L}(\mathfrak{g}))$ may not be semisimple, there is the following partial remedy for its structure.

Prop. 2.7. *The category $\mathbf{Rep}(\mathfrak{L}(\mathfrak{g}))$ has a unique decomposition into a direct sum of indecomposable, abelian subcategories*

Proof. We say that two indecomposable objects $V_1, V_2 \in \text{Obj}(\mathbf{Rep}(\mathfrak{L}(\mathfrak{g})))$ are linked if $\mathbf{Rep}(\mathfrak{L}(\mathfrak{g}))$ admits no splitting into two abelian categories $\mathbf{C}_1$ and $\mathbf{C}_2$ such that $V_1 \in \text{Obj}(\mathbf{C}_1)$ and $V_2 \in \text{Obj}(\mathbf{C}_2)$. Let I be the index set of equivalence classes of linked indecomposable objects in $\mathbf{Rep}(\mathfrak{L}(\mathfrak{g}))$. For any $i \in I$, let $\mathbf{C}_i$ be the subcategory of $\mathbf{Rep}(\mathfrak{L}(\mathfrak{g}))$ containing as objects direct sums of the objects in equivalence class i . Since the decomposition of any abelian tensor category into its indecomposables is unique, we have

$$\mathbf{Rep}(\mathfrak{L}(\mathfrak{g})) = \bigoplus_{i \in I} \mathbf{C}_i.$$

We claim that the $\mathbf{C}_i$ are themselves indecomposable. For, if there existed a non-trivial decomposition $\mathbf{C}_i = \mathbf{C}_i^1 \oplus \mathbf{C}_i^2$, then any indecomposable objects $V_1 \in \text{Obj}(\mathbf{C}_i^1)$ and $V_2 \in \text{Obj}(\mathbf{C}_i^2)$ cannot be linked. However, this contradicts the definition of $\mathbf{C}_i$. The claim follows. Uniqueness of the decomposition follows from its definition via equivalence classes of linked objects, whence any other, non-isomorphic decomposition necessarily cannot be a decomposition into indecomposables. $\square$

The subcategories $\mathbf{C}_i$ constructed in prop. 2.7 are called blocks of $\mathbf{Rep}(\mathfrak{L}(\mathfrak{g}))$. Likewise, the decomposition is called block decomposition. In more generality,

prop. 2.7 holds for any abelian category, as the proof only used that property. We turn to a construction of further finite-dimensional $\mathfrak{L}(\mathfrak{g})$ -modules. Recall the triangular decomposition $\mathfrak{L}(\mathfrak{g}) = \mathfrak{L}(\mathfrak{n}_+) \oplus \mathfrak{L}(\mathfrak{h}) \oplus \mathfrak{L}(\mathfrak{n}_-)$ from cor. 1.19 with respect to the triangular decomposition $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$.

Def. 2.8. An $\mathfrak{L}(\mathfrak{g})$ -module $V(\lambda)$ is called a highest weight module with highest weight $\lambda \in \mathfrak{L}(\mathfrak{h})^*$ if there exists $v_\lambda \in V(\lambda)$ with $\mathfrak{U}(\mathfrak{L}(\mathfrak{g}))v_\lambda = V$ such that

$$\mathfrak{L}(\mathfrak{n}_+)v_\lambda = 0 \quad \text{and} \quad \forall h \in \mathfrak{L}(\mathfrak{h}) : hv_\lambda = \lambda(h)v_\lambda.$$

We call v_λ a highest weight vector.

To define a class of highest weight $\mathfrak{L}(\mathfrak{g})$ -modules that will be of interest to us, we need the notion of a Drinfeld polynomial.

Def. 2.9. Let $n = \text{rank}(\mathfrak{g})$ and $I = \{1, \dots, n\}$. Define the monoid of Drinfeld polynomials $\mathcal{P}$ as the monoid of n -tuples of $\mathbb{C}$ -polynomials $\pi = (\pi_1, \dots, \pi_n) \in (\mathbb{C}[u])^n$ with u an indeterminate such the constant term of each π_i is 1. Denoting by P_+ the positive weight lattice of $\mathfrak{g}$, we associate to each $i \in I$, $\lambda \in P_+$, and $a \in \mathbb{C}^\times$, a unique element $\pi_{\lambda,a} \in \mathcal{P}$, given by

$$\pi_{\lambda,a} = ((1-au)^{\lambda(h_i)}) := ((1-au)^{\lambda(h_1)}, \dots, (1-au)^{\lambda(h_n)}),$$

where $h_i \in \mathfrak{h}$ are the Cartan generators of $\mathfrak{g}$. Finally, denoting the fundamental weights of $\mathfrak{g}$ by ω_i , we define a map

$$\Lambda : \mathcal{P} \rightarrow P_+; \pi \mapsto \lambda_\pi := \sum_{i \in I} \deg_{\mathbb{C}[u]}(\pi_i) \omega_i.$$

Lem. 2.10. Any $\pi \in \mathcal{P}$ decomposes uniquely into a product

$$\pi = \prod_{j=1}^m \pi_{\lambda_j, a_j},$$

where $\lambda_j \in P_+$ and all $a_j \in \mathbb{C}^\times$ are pairwise distinct.

Proof. Since every polynomial over $\mathbb{C}$ factors uniquely into irreducibles, every π_i must factor into a unique combination of irreducibles with constant terms 1. We may, then, factor π as a product of polynomials of the form

$$\pi_{i,a} = ((1-au)^{\delta_{ij}}) = (0, \dots, 0, (1-au), 0, \dots, 0)$$

with the only non-zero term being in the i^{th} coordinate. Collecting multiplicities $\lambda(h_i)$, we may, therefore, conclude the claim. $\square$

We call the decomposition in lem. 2.10 the standard decomposition of $\pi \in \mathcal{P}$. In this standard decomposition, the scalars $a_i \in \mathbb{C}^\times$ are called the coordinates of π . Using the Drinfeld polynomials, we define a class of finite-dimensional $\mathfrak{L}(\mathfrak{g})$ -modules that take the role analogous to that of Verma modules in the classical, finite-dimensional representation theory.

Def. 2.11. Let $\pi \in \mathcal{P}$ with standard decomposition $\pi = \prod_{j=1}^m \pi_{\lambda_j, a_j}$ as in lem. 2.10. Furthermore, let $J_\pi \trianglelefteq \mathfrak{U}(\mathfrak{L}(\mathfrak{g}))$ be the ideal generated by

$$J_\pi = \left\langle \mathfrak{L}(\mathfrak{n}_+), (f_i^{\lambda_\pi(h_i)+1})_{i \in I}, (h \otimes t^k - \sum_{j=1}^m a_j^k \lambda_j(h))_{h \in \mathfrak{h}, k \in \mathbb{Z}} \right\rangle.$$

Define the Weyl module $W(\pi)$ as the $\mathfrak{L}(\mathfrak{g})$ -module defined by the quotient

$$W(\pi) = \mathfrak{U}(\mathfrak{L}(\mathfrak{g}))/J_\pi.$$

Furthermore, define $w_\pi := \text{pr}(1)$, where $\text{pr} : \mathfrak{U}(\mathfrak{L}(\mathfrak{g})) \rightarrow W(\pi)$ denotes the canonical projection.

Prop. 2.12. *For any $\pi \in \mathcal{P}$, the Weyl module $W(\pi)$ is a finite-dimensional, highest weight $\mathfrak{L}(\mathfrak{g})$ -module.*

Proof. The proof is immediate. Taking the ideal J_π from def. 2.11, the first and third class of generators ensure that $W(\pi)$ is a highest weight $\mathfrak{L}(\mathfrak{g})$ -module, whereas the second class of generators ensures finite-dimensionality. $\square$

The Weyl modules carry a very desirable structure under the tensor product of modules. This notion is formalised in the following results.

Thm. 2.13. *Let $\pi, \bar{\pi} \in \mathcal{P}$ be such that π_i and $\bar{\pi}_j$ are coprime for all $1 \leq i, j \leq n$. Then*

$$W(\pi\bar{\pi}) = W(\pi) \otimes W(\bar{\pi}).$$

Proof. [Chari & Pressley 2001, thm. 2]. $\square$

Cor. 2.14. *Let $\pi \in \mathcal{P}$ with standard decomposition $\pi = \prod_{j=1}^m \pi_{\lambda_j, a_j}$. Then*

$$W(\pi) = \bigotimes_{j=1}^m W(\pi_{\lambda_j, a_j})$$

as $\mathfrak{L}(\mathfrak{g})$ -modules.

The following implications clarify the analogy between Weyl modules and Verma modules.

Prop. 2.15. *Let $\pi, \bar{\pi} \in \mathcal{P}$. There exists a unique irreducible quotient $\mathfrak{L}(\mathfrak{g})$ -module $V(\pi)$ of $W(\pi)$. Conversely, every finite-dimensional, irreducible $\mathfrak{L}(\mathfrak{g})$ -module may be constructed as such a quotient with the property that for $\pi \neq \bar{\pi}$ the quotient modules $V(\pi), V(\bar{\pi})$ are non-isomorphic.*

Prop. 2.16. *Suppose $V \in \text{Obj}(\mathbf{Rep}(\mathfrak{L}(\mathfrak{g})))$ is a highest weight module. Then there exists $\pi \in \mathcal{P}$ such that V is a quotient $\mathfrak{L}(\mathfrak{g})$ -module of $W(\pi)$.*

Proof of props. 2.15 and 2.16. [Chari & Pressley 2001, prop. 2.1]. $\square$

Therefore, Weyl modules entirely classify highest weight $\mathfrak{L}(\mathfrak{g})$ -modules. Thereby, they give an explicit construction of a class of important and geometrically intuitive objects in $\mathbf{Rep}(\mathfrak{L}(\mathfrak{g}))$. Even more, prop. 2.15 states that the isomorphism classes of highest weight $\mathfrak{L}(\mathfrak{g})$ -modules are in bijective correspondence with Drinfeld polynomials $\mathcal{P}$. Therefore, we may parametrise highest weight $\mathfrak{L}(\mathfrak{g})$ -modules $V(\pi)$ by Drinfeld polynomials $\pi \in \mathcal{P}$. To complete the categorical classification of $\mathfrak{L}(\mathfrak{g})$ -modules, we describe the block decomposition of $\mathbf{Rep}(\mathfrak{L}(\mathfrak{g}))$ from prop. 2.7. To this end, we define the monoid of spectral characters of $\mathfrak{L}(\mathfrak{g})$.

Def. 2.17. Let P be the weight space and Δ the root space of $\mathfrak{g}$. For any $a \in \mathbb{C}^\times$, denote by δ_a the characteristic function of a . Define the monoid Ξ of spectral characters of $\mathfrak{L}(\mathfrak{g})$ as the monoid of finitely supported functions $\chi : \mathbb{C}^\times \rightarrow P/\Delta$. Furthermore, for a positive weight $\lambda \in P_+$ and $a \in \mathbb{C}^\times$, define

$$\chi_{\lambda,a} = \delta_a \bar{\lambda},$$

where $\bar{\lambda}$ is the canonical projection of λ onto P/Δ .

Def. 2.18. Let $\pi \in \mathcal{P}$ be a Drinfeld polynomial with standard decomposition $\pi = \prod_{j=1}^m \pi_{\lambda_j, a_j}$. Define the spectral character $\chi_\pi \in \Xi$ of π

$$\chi_\pi = \sum_{j=1}^m \chi_{\lambda_j, a_j}$$

with respect to the standard decomposition.

The map $\pi \mapsto \chi_\pi$ is a homomorphism from a multiplicative monoid $\mathcal{P}$ to an additive monoid Ξ . Therefore, the spectral characters from def. 2.18 are additive. It holds that

$$\chi_{\pi_1 \pi_2} = \chi_{\pi_1} + \chi_{\pi_2}$$

for arbitrary Drinfeld polynomials $\pi_1, \pi_2 \in \mathcal{P}$.

Def. 2.19. An element $V \in \text{Obj}(\mathbf{Rep}(\mathfrak{L}(\mathfrak{g})))$ is defined to have spectral character $\chi \in \Xi$ if for every irreducible component $V(\pi)$ of V it holds that $\chi = \chi_\pi$ for the spectral character χ_π of the parametrising Drinfeld polynomial $\pi \in \mathcal{P}$. Denote by $\mathbf{C}_\chi$ the abelian subcategory of $\mathbf{Rep}(\mathfrak{L}(\mathfrak{g}))$ that contains all objects with spectral character χ .

The content of def. 2.19 may be formalised more rigorously as follows. By prop. 2.6, $\mathbf{Rep}(\mathfrak{L}(\mathfrak{g}))$ is not semisimple and, therefore, does not generally admit a unique decomposition series. However, since $\mathbf{Rep}(\mathfrak{L}(\mathfrak{g}))$ is an abelian tensor category, every object $V \in \text{Obj}(\mathbf{Rep}(\mathfrak{L}(\mathfrak{g})))$ admits a Jordan-Hölder series of finite length. The elements of this series are precisely the required irreducible components. Since, by prop. 2.15, any such irreducible component is parametrised by a Drinfeld polynomial, the remaining claims follow. We now work towards a classification of the block decomposition of $\mathbf{Rep}(\mathfrak{L}(\mathfrak{g}))$.

Lem. 2.20. Let $\chi_1, \chi_2 \in \xi$, $V(\pi_1) \in \text{Obj}(\mathbf{C}_{\chi_1})$, and $V(\pi_2) \in \text{Obj}(\mathbf{C}_{\chi_2})$ for some $\pi_1, \pi_2 \in \mathcal{P}$. Then $V(\pi_1) \otimes V(\pi_2) \in \text{Obj}(\mathbf{C}_{\chi_1 + \chi_2})$.

Proof. By cor. 2.14, there exist $\lambda_j \in P_+$ and pair-wise distinct $a_j \in \mathbb{C}^\times$ such that

$$V(\pi_1) = \bigotimes_{j=1}^m V(\pi_{\lambda_j, a_j}).$$

Analogously

$$V(\pi_2) = \bigotimes_{j=1}^l V(\pi_{\mu_j, b_j})$$

for $\mu_j \in P_+$ and $b_j \in \mathbb{C}^\times$ pair-wise distinct. Set $S = \{a_1, \dots, a_m\} \cap \{b_1, \dots, b_l\}$. We prove the claim by induction on the cardinality of S . If S is empty, $V(\pi_1) \otimes V(\pi_2)$ is irreducible and the claim follows from the additivity of the spectral character on irreducibles. Therefore, suppose S is not empty. After reordering, we may assume $a_1 = b_1$. We take the Clebsch-Gordan decomposition

$$V(\pi_{\lambda_1, a_1}) \otimes V(\pi_{\mu_1, a_1}) = \bigoplus_{\nu \in P_+} V(\pi_{\nu, a_1})^{\oplus m_\nu},$$

where m_ν denote the multiplicities of the factors in the decomposition. By virtue of the Clebsch-Gordan decomposition, $\lambda_1 + \mu_1 - \nu$ must be in the root space. However, by def. 2.17, the spectral character of any root must equate to zero. Therefore, $\chi_{\nu, a_1} = \chi_{\lambda_1, a_1} + \chi_{\mu_1, a_1}$ for all $\nu \in P_+$ in the Clebsch-Gordan decomposition. Factoring this decomposition out of the tensor product, we find

$$V(\pi_1) \otimes V(\pi_2) = \left(\bigoplus_{\nu \in P_+} V(\pi_{\nu, a_1})^{\oplus m_\nu} \right) \otimes \left(\bigotimes_{j=2}^m V(\pi_{\lambda_j, a_j}) \otimes \bigotimes_{j=2}^l V(\pi_{\mu_j, b_j}) \right).$$

By the above, the first term satisfies the claim, whereas the induction hypothesis ensures likewise for the second term. We conclude the claim by induction. $\square$

The following results are central in the classification of the block decomposition.

Prop. 2.21. *Let $\chi \in \Xi$. Any two irreducible objects in $\mathbf{C}_\chi$ are linked.*

Proof. [Chari & Moura 2004, prop. 2.3]. $\square$

Prop. 2.22. *Any indecomposable object in $\mathbf{Rep}(\mathfrak{V}(\mathfrak{g}))$ admits a spectral character.*

Proof. [Chari & Moura 2004, prop. 2.4]. $\square$

Combining the results of props. 2.21 and 2.22, we find the desired block decomposition. Thereby, we give the explicit form of the result in prop. 2.7.

Thm. 2.23. *The category $\mathbf{Rep}(\mathfrak{V}(\mathfrak{g}))$ admits the block decomposition*

$$\mathbf{Rep}(\mathfrak{V}(\mathfrak{g})) = \bigoplus_{\chi \in \Xi} \mathbf{C}_\chi,$$

where each $\mathbf{C}_\chi$ is a block. In particular, the blocks of $\mathbf{Rep}(\mathfrak{V}(\mathfrak{g}))$ are in bijective correspondence with spectral characters Ξ of $\mathfrak{V}(\mathfrak{g})$.

Proof. By prop. 2.22, any indecomposable object of $\mathbf{Rep}(\mathfrak{V}(\mathfrak{g}))$ is in $\mathbf{C}_\chi$ for some $\chi \in \Xi$, ensuring the orthogonality of the decomposition into indecomposables. It follows, then, from prop. 2.21 that the $\mathbf{C}_\chi$ indeed form blocks. $\square$

The statement of thm. 2.23 completes the structure classification of $\mathbf{Rep}(\mathfrak{V}(\mathfrak{g}))$. To conclude, we classify the finite-dimensional representations of the current algebra $\tilde{\mathfrak{V}}(\mathfrak{g}) = \mathfrak{V}(\mathfrak{g}) \oplus \mathbb{C}K$ in terms of those of the underlying loop algebra. This also gives a classification of the finite-dimensional $\hat{\mathfrak{V}}(\mathfrak{g})$ -modules, since $\hat{\mathfrak{V}}(\mathfrak{g}) = \tilde{\mathfrak{V}}(\mathfrak{g}) \rtimes \mathbb{C}d$ is only a vector space extension of the current algebra by an exterior derivation d . Clearly, any $\mathfrak{V}(\mathfrak{g})$ -module can be turned into a $\tilde{\mathfrak{V}}(\mathfrak{g})$ -module by setting the action of the central element K to zero. Furthermore, considering the finite-dimensional representations $\mathbf{Rep}(\mathfrak{V}(\mathfrak{g}))$ of the loop algebra also gives a classification of the finite-dimensional representations of the corresponding current algebra $\tilde{\mathfrak{V}}(\mathfrak{g})$.

Prop. 2.24. *There is an equivalence of categories $\mathbf{Rep}(\tilde{\mathfrak{V}}(\mathfrak{g})) \cong \mathbf{Rep}(\mathfrak{V}(\mathfrak{g}))$.*

Proof. We want to show that the center of $\tilde{\mathfrak{V}}(\mathfrak{g}) = \mathfrak{V}(\mathfrak{g}) \oplus \mathbb{C}K$ acts trivially on any finite-dimensional representation. Let $\varrho : \tilde{\mathfrak{V}}(\mathfrak{g}) \rightarrow \text{End}(V)$ be a finite-dimensional representation. Since ϱ is a Lie algebra homomorphism, $\ker(\varrho) \trianglelefteq \tilde{\mathfrak{V}}(\mathfrak{g})$ is an ideal. By the first Lie algebra isomorphism theorem, $\tilde{\mathfrak{V}}(\mathfrak{g})/\ker(\varrho)$ is necessarily finite-dimensional. A general element in $\ker(\varrho) \setminus \{0\}$ is of the form

$$\sum_{i \in I} (x_i \otimes P_i) + \mu_1 K,$$

where $I \subset \mathbb{N}$ is an index set, $x_i \in \mathfrak{g}_{\alpha_i}$, $P_i \in \mathfrak{Q}$ and $\mu_1 \in \mathbb{C}$. We prove the claim by induction on the number of non-zero, non-central terms $x_i \otimes P_i$. If there are no such terms, then by necessity $K \in \ker(\varrho)$. This proves the base case. For a non-zero term $x_j \otimes P_j$ with $x_j \in \mathfrak{g}_{\alpha_j}$ there exists, by properties of standard the inner pairing on a semisimple Lie algebra $\mathfrak{g}$, some $y_j \in \mathfrak{g}_{-\alpha_j}$ such that $\langle y_j, x_j \rangle \neq 0$ but $\langle y_j, x_k \rangle = 0$ for all other pairings. There exists $Q_j \in \mathfrak{Q}$ such that

$$\mu_2 := \text{Res}_{t=0}(\langle y_j \otimes dQ_j, x_j \otimes P_j \rangle) \neq 0,$$

where d denotes the exterior derivation. Computing the Lie bracket, we find

$$[y_j \otimes Q_j, \sum_{i \in I} (x_i \otimes P_i) + \mu_1 K] = \sum_{i \in I} ([y_j, x_i] \otimes Q_j P_i) + \mu_2 K.$$

If $[y_j, x_j] = 0$ we are done by the induction hypothesis. Otherwise, by the grading on the root decomposition, $[y_j, x_j] \in \mathfrak{h}$. Since the inner pairing is completely non-degenerate on $\mathfrak{h}$, there exists $h \in \mathfrak{h}$ such that $\langle h, [y_j, x_j] \rangle \neq 0$. Let $R \in \mathfrak{Q}$ such that

$$\mu_3 := \text{Res}_{t=0}(\langle h \otimes dR, [y_j, x_j] \otimes Q_j P_j \rangle) \neq 0.$$

Then

$$[h \otimes R, \sum_{i \in I} ([y_j, x_i] \otimes Q_j P_i) + \mu_2 K] = \sum_{i \in I} ([h, [y_j, x_i]] \otimes R Q_j P_i) + \mu_3 K.$$

Since $\mathfrak{h}$ is an abelian subalgebra, $[h, [y_j, x_j]] = 0$. Therefore, we conclude the claim by the induction hypothesis. $\square$

3 Representation Theory of Twisted Loop Algebras $\mathfrak{L}(\mathfrak{g}, \psi)$

We characterise $\mathfrak{L}(\mathfrak{g}, \psi)$ -representations. Recall that, by prop. 1.13, we may assume ψ to be a diagram automorphism of some order $r \in \mathbb{N}$. Since $\mathfrak{L}(\mathfrak{g}, \psi) \leq \mathfrak{L}(\mathfrak{g})$ is a Lie subalgebra by lem. 1.4, any $\mathfrak{L}(\mathfrak{g})$ -module canonically restricts to a $\mathfrak{L}(\mathfrak{g}, \psi)$ -representation. Furthermore, in the grading of equations (1.1) and (1.2), $\mathfrak{g}_0$ is the ψ -invariant subalgebra of $\mathfrak{L}(\mathfrak{g}, \psi)$. By cor. 1.14, $\mathfrak{g}_0$ is simple. In particular, by the canonical inclusion

$$\mathfrak{g}_0 \hookrightarrow \mathfrak{L}(\mathfrak{g}, \psi); x \mapsto x \otimes 1$$

any $\mathfrak{L}(\mathfrak{g}, \psi)$ -module restricts to a $\mathfrak{g}_0$ -representation. In complete analogy to the untwisted representation theory, we consider the following category of representations.

Def. 3.1. Let $\mathbf{Rep}(\mathfrak{L}(\mathfrak{g}, \psi))$ be the category of finite-dimensional $\mathfrak{L}(\mathfrak{g}, \psi)$ -modules, whose objects are finite-dimensional representations of $\mathfrak{L}(\mathfrak{g}, \psi)$ and whose morphisms are equivariant maps.

There is the following immediate implication from prop. 2.6.

Cor. 3.2. *The category $\mathbf{Rep}(\mathfrak{L}(\mathfrak{g}, \psi))$ is not semisimple.*

Similarly, the following is an analogue of prop. 2.7.

Cor. 3.3. *$\mathbf{Rep}(\mathfrak{L}(\mathfrak{g}, \psi))$ has a unique decomposition into a direct sum of indecomposable, abelian subcategories, called block decomposition of $\mathbf{Rep}(\mathfrak{L}(\mathfrak{g}, \psi))$.*

Highest weight $\mathfrak{L}(\mathfrak{g}, \psi)$ -modules are defined in accordance with def. 2.8. The analogy to the untwisted representation theory can be driven even further by defining the monoid of twisted Drinfeld polynomials and using these to construct the irreducible Weyl modules of $\mathfrak{L}(\mathfrak{g}, \psi)$. However, it turns out that among the simple Lie algebras with non-trivial diagram automorphisms, those of type A_{2n} are subject to a peculiar constraint arising from the structure of its $\mathfrak{sl}_2$ -subalgebras. This peculiarity necessitates a careful treatment, which we formalise in the following. Let $n = \text{rank}(\mathfrak{g}_0)$ and $I_0 = \{1, \dots, n\}$. If $\mathfrak{g}$ is not of type A_{2n} , we define, for $i \in I_0$, elements $h_{i, \epsilon} \in \mathfrak{g}_\epsilon$ by

$$h_{i, \epsilon} = |\text{Stab}_G(i)|^{-1} \sum_{j=0}^{r-1} \xi^{j\epsilon} h_{\psi^j(i)},$$

where G is the cyclic group with elements ψ^j and $h_k \in \mathfrak{h}_0$ are the Chevalley generators of the Cartan subalgebra $\mathfrak{h}_0 \leq \mathfrak{g}$. In the peculiar case when $\mathfrak{g}$ is of type A_{2n} , we have a unique diagram automorphism of order 2. Then, we set

$$h_{n,0} = 2(h_n + h_{n+1}) \quad \text{and} \quad h_{n,1} = h_n - h_{n+1},$$

keeping the remaining cases in analogy to the general case.

Rem. 3.4. We formalise the constraint arising in the representation theory of $\mathfrak{Q}(\mathfrak{g}, \psi)$ when $\mathfrak{g}$ is of type A_{2n} . Denoting by e_n, f_n the usual Chevalley operators of $\mathfrak{g}_0$, define

$$f_{n,0} = \sqrt{2}(f_n + f_{n+1}) \quad \text{and} \quad f_{n,1} = -\sqrt{2}(f_n - f_{n+1}).$$

Furthermore, we define

$$f_{n,1}^- = \frac{1}{4}[f_{n,0}, f_{n,1}],$$

and, analogously

$$e_{n,1}^+ = -\frac{1}{4}[e_{n,0}, e_{n,1}].$$

The operator $f_{n,1}^-$ must act nilpotently on any highest weight vector in a highest weight $\mathfrak{Q}(\mathfrak{g}, \psi)$ -module. However, the $\mathfrak{sl}_2$ -subalgebra spanned by this operator is given by

$$\mathfrak{sl}_2 \cong \langle e_{n,1}^+ \otimes t^{-1}, \frac{1}{2}h_{n,0}, f_{n,1}^- \otimes t \rangle.$$

Therefore, by the usual representation theory of $\mathfrak{sl}_2$, it must hold that $\lambda(\frac{1}{2}h_{n,0}) \in \mathbb{Z}$ for the highest weight λ . Due to this scaling, it becomes necessary to consider Lie algebras of type A_{2n} separately.

Instead of, as per the usual procedure, considering the positive weight lattice of A_{2n} to classify highest weight representations, we define the following modified weight lattice.

Def. 3.5. Denote by $(P_0)_+$ the positive weight lattice of $\mathfrak{g}_0$. Define the subset $(P_\psi)_+ \subset (P_0)_+$ as

$$(P_\psi)_+ = \begin{cases} (P_0)_+ & \text{if } \mathfrak{g} \text{ not of type } A_{2n}, \\ \{\lambda \in (P_0)_+ \mid \lambda(h_{n,0}) \in 2\mathbb{Z}\} & \text{if } \mathfrak{g} \text{ of type } A_{2n}, \end{cases}$$

where $h_i \in \mathfrak{h}_0$ are the Chevalley generators of the Cartan subalgebra $\mathfrak{h}_0 \leq \mathfrak{g}_0$.

Def. 3.6. Associate to each $i \in I_0$, $\lambda \in (P_\psi)_+$, and $a \in \mathbb{C}^\times$ a unique n -tuple of $\mathbb{C}$ -polynomials $\tilde{\pi}_{\lambda,a} \in (\mathbb{C}[u])^n$ with constant term 1, given by

$$\tilde{\pi}_{\lambda,a} = \begin{cases} \prod_{i \in I_0} ((1 - a^{\langle \alpha_i, \alpha_i \rangle})^{\delta_{ij}})^{\lambda(h_i)} & \text{if } \mathfrak{g} \text{ not of type } A_{2n}, \\ \prod_{i \in I_0} ((1 - a)^{\delta_{ij}})^{(1 - \frac{1}{2}\delta_{in})\lambda(h_i)} & \text{if } \mathfrak{g} \text{ of type } A_{2n}, \end{cases}$$

where $h_i \in \mathfrak{h}_0$ are in the Cartan subalgebra $\mathfrak{h}_0 \leq \mathfrak{g}_0$ and α_i are the simple roots of $\mathfrak{g}_0$. Define the monoid of twisted Drinfeld polynomials $\tilde{\mathcal{P}}$ as the monoid of n -tuples of

$\mathbb{C}$ -polynomials generated by elements $\tilde{\pi}_{\lambda,a}$. Denoting by ω_i the fundamental roots of $\mathfrak{g}_0$, we define a map

$$\Lambda : \tilde{\mathcal{P}} \rightarrow (P_\psi)_+; \tilde{\pi} \mapsto \begin{cases} \sum_{i \in I_0} \deg_{\mathbb{C}[u]}(\tilde{\pi}) \omega_i & \text{if } \mathfrak{g} \text{ not of type } A_{2n}, \\ \sum_{i \in I_0} (1 + \delta_{in}) \deg_{\mathbb{C}[u]}(\tilde{\pi}) \omega_i & \text{if } \mathfrak{g} \text{ of type } A_{2n}. \end{cases}$$

Twisted Drinfeld polynomials also admit decompositions into the unique elements $\tilde{\pi}_{\lambda,a}$ as given in def. 3.6.

Lem. 3.7. *Any twisted Drinfeld polynomial $\tilde{\pi} \in \tilde{\mathcal{P}}$ admits a decomposition*

$$\tilde{\pi} = \prod_{j=1}^m \prod_{\epsilon=0}^{r-1} \tilde{\pi}_{\lambda_{j,\epsilon}, \xi^\epsilon a_j}$$

for some $m \in \mathbb{N}$ with pair-wise distinct $\lambda_{j,\epsilon} \in (P_\psi)_+$ and coordinates $a_j \in \mathbb{C}^\times$ that are pair-wise distinct and whose powers a_j^r are pair-wise distinct.

Proof. Similarly to the untwisted case in lem. 2.10, the proof is immediate by collecting multiplicities of

$$\tilde{\pi}_{i,a} = \begin{cases} (1 - a^{\langle \alpha_i, \alpha_i \rangle})^{\delta_{ij}} & \text{if } \mathfrak{g} \text{ not of type } A_{2n}, \\ (1 - a)^{\delta_{ij}} & \text{if } \mathfrak{g} \text{ of type } A_{2n}, \end{cases}$$

noting that the additional degrees of freedom given by the ξ^ϵ allow us to respect the non-trivial eigenspace decomposition from equations (1.1) and (1.2) that arises from the twisting. The powers a_j^r of the coordinates are guaranteed to be pair-wise distinct by the eigenspace grading. $\square$

Note that the decomposition guaranteed by lem. 3.7 for $\tilde{\pi} \in \tilde{\mathcal{P}}$ need not necessarily be unique. We call any such decomposition a standard decomposition of $\tilde{\pi}$. We now define the class of twisted Weyl modules.

Def. 3.8. Let $\tilde{\pi} \in \tilde{\mathcal{P}}$ with a standard decomposition $\tilde{\pi} = \prod_{j=1}^m \prod_{\epsilon=0}^{r-1} \tilde{\pi}_{\lambda_{j,\epsilon}, \xi^\epsilon a_j}$ as in lem. 3.7. Define $J_{\tilde{\pi}} \trianglelefteq \mathfrak{U}(\mathfrak{g}, \psi)$ to be the ideal generated by

$$J_{\tilde{\pi}} = \begin{cases} \left\langle \mathfrak{L}(\mathfrak{n}_+, \psi), f_i^{\lambda_{\tilde{\pi}}(h_i)+1}, h_{i,\epsilon} \otimes t^{rk-\epsilon} - \sum_{j=1}^m a_j^{rk-\epsilon} \lambda_j(h_{i,0}) \right\rangle, & \mathfrak{g} \neq A_{2n}, \\ \left\langle \mathfrak{L}(\mathfrak{n}_+, \psi), f_i^{\lambda_{\tilde{\pi}}(h_i)+1}, h_{i,\epsilon} \otimes t^{rk-\epsilon} - \sum_{j=1}^m (1 - \frac{1}{2} \delta_{in}) a_j^{rk-\epsilon} \lambda_j(h_{i,0}) \right\rangle, & \mathfrak{g} = A_{2n}. \end{cases}$$

Define the Weyl module $W(\tilde{\pi})$ as the $\mathfrak{L}(\mathfrak{g}, \psi)$ -module defined by the quotient

$$W(\tilde{\pi}) = \mathfrak{U}(\mathfrak{g}, \psi) / J_{\tilde{\pi}}.$$

Furthermore, define $w_{\tilde{\pi}} := \text{pr}(1)$, where $\text{pr} : \mathfrak{U}(\mathfrak{g}, \psi) \rightarrow W(\tilde{\pi})$ denotes the canonical projection.

In analogy to prop. 2.12, the proof that $W(\tilde{\pi})$ is finite-dimensional follows immediately from the definition. For, the quotient ideal that generates the module ensures local nilpotency of all the generators of the resulting twisted Weyl module. The following results are, then, the analogues to props. 2.15 and 2.16.

Prop. 3.9. *Let $\tilde{\pi}, \tilde{\pi}' \in \tilde{\mathcal{P}}$. The twisted Weyl module $W(\tilde{\pi})$ has a unique irreducible quotient $\mathfrak{L}(\mathfrak{g}, \psi)$ -module $V(\tilde{\pi})$ of $W(\tilde{\pi})$. Conversely, every finite-dimensional, irreducible $\mathfrak{L}(\mathfrak{g}, \psi)$ -module may be constructed as such a quotient with the property that, for $\tilde{\pi} \neq \tilde{\pi}'$, the $\mathfrak{L}(\mathfrak{g}, \psi)$ -modules $V(\tilde{\pi})$ and $V(\tilde{\pi})'$ are non-isomorphic.*

Prop. 3.10. *Suppose $V \in \text{Obj}(\mathbf{Rep}(\mathfrak{L}(\mathfrak{g}, \psi)))$ is a highest weight module. Then there exists $\tilde{\pi} \in \tilde{\mathcal{P}}$ such that V is a quotient $\mathfrak{L}(\mathfrak{g}, \psi)$ -module of $W(\tilde{\pi})$.*

Proof of props. 3.9 and 3.10. [Chari, Fourier & Senesi 2008, 3.7]. $\square$

Akin to the untwisted case in cor. 2.14, twisted Weyl modules admit a product decomposition with respect to any standard decomposition of their corresponding Drinfeld polynomial.

Prop. 3.11. *Let $\tilde{\pi} \in \tilde{\mathcal{P}}$ with a standard decomposition $\tilde{\pi} = \prod_{j=1}^m \prod_{\epsilon=0}^{r-1} \tilde{\pi}_{\lambda_{j,\epsilon}, \xi^\epsilon a_j}$. Then*

$$W(\tilde{\pi}) \cong \bigotimes_{j=1}^m W\left(\prod_{\epsilon=0}^{r-1} \tilde{\pi}_{\lambda_{j,\epsilon}, \xi^\epsilon a_j}\right).$$

Proof. [Chari, Fourier & Senesi 2008, thm. 2]. $\square$

There is a close relationship between the untwisted $\mathfrak{L}(\mathfrak{g})$ -modules and twisted $\mathfrak{L}(\mathfrak{g}, \psi)$ -modules, which arises from the structure between untwisted and twisted Drinfeld polynomials. Denote by ω_i the fundamental weights of $\mathfrak{g}$ and by I their index set. Take some $\lambda = \sum_{i \in I} m_i \omega_i \in P_+$ for multiplicities $m_i \in \mathbb{N}$. For each different order r of diagram automorphism and type of $\mathfrak{g}$, we define specific elements $\lambda(\epsilon) \in (P_\psi)_+$ for $\epsilon \in \{0, \dots, r-1\}$ as follows. Suppose first that $r = 2$ and that $\mathfrak{g}$ is not of type A_{2n} . In this case, we define

$$\lambda(0) = \sum_{i \in I_0} m_i \omega_i, \quad \text{and} \quad \lambda(1) = \sum_{\substack{i \in I_0 \\ \psi(i) \neq i}} m_{\psi(i)} \omega_i.$$

For $r = 2$ and $\mathfrak{g}$ of type A_{2n} , we define

$$\lambda(0) = \sum_{i \in I_0} (1 + \delta_{in}) m_i \omega_i, \quad \text{and} \quad \lambda(1) = \sum_{\substack{i \in I_0 \\ \psi(i) \neq i}} (1 + \delta_{\psi(i)n}) m_{\psi(i)} \omega_i.$$

Lastly, for the case of $r = 3$, we set

$$\lambda(0) = m_1\omega_1 + m_2\omega_2, \quad \lambda(1) = m_3\omega_2, \quad \text{and} \quad \lambda(2) = m_4\omega_1.$$

These weights $\lambda(\epsilon)$ are central to describing the connection between untwisted and twisted Drinfeld polynomials.

Def. 3.12. Let $\pi \in \mathcal{P}$ be an untwisted Drinfeld polynomial with unique standard decomposition $\pi = \prod_{j=1}^m \pi_{\lambda_j, a_j}$ as guaranteed by lem. 2.10. Define the restriction map

$$\text{res} : \mathcal{P} \rightarrow \tilde{\mathcal{P}}; \pi \mapsto \prod_{j=1}^m \prod_{\epsilon=0}^{r-1} \tilde{\pi}_{\lambda_j(\epsilon), \xi^\epsilon a_j}.$$

Furthermore, for a twisted Drinfeld polynomial $\tilde{\pi} \in \tilde{\mathcal{P}}$, define the set of induced Drinfeld polynomials

$$\text{ind} : \tilde{\mathcal{P}} \rightarrow \mathbb{P}(\mathcal{P}); \tilde{\pi} \mapsto \{\pi \in \mathcal{P} \mid \text{res}(\pi) = \tilde{\pi}\},$$

where $\mathbb{P}(\mathcal{P})$ denotes the power set of $\mathcal{P}$.

The set of induced Drinfeld polynomials turns out to be the central ingredient in describing the connection between untwisted and twisted Weyl modules. The main result pertaining to this connection is the following.

Thm. 3.13. *Let $\tilde{\pi} \in \tilde{\mathcal{P}}$. For $\pi \in \text{ind}(\tilde{\pi})$, it holds that*

$$W(\tilde{\pi}) \cong W(\pi) \Big|_{\mathfrak{L}(\mathfrak{g}, \psi)}.$$

Proof. [Chari, Fourier & Senesi 2008, thm. 2]. □

Cor. 3.14. *Let $\tilde{\pi} \in \tilde{\mathcal{P}}$ and $\pi \in \text{ind}(\tilde{\pi})$. For the irreducible quotients $V(\tilde{\pi})$ of $W(\tilde{\pi})$ and $V(\pi)$ of $W(\pi)$ it holds that*

$$V(\tilde{\pi}) \cong V(\pi) \Big|_{\mathfrak{L}(\mathfrak{g}, \psi)}.$$

Taken together, props. 3.9 and 3.10 and cor. 3.14 classify the irreducible $\mathfrak{L}(\mathfrak{g}, \psi)$ -modules. Furthermore, we have constructed a large general class of finite-dimensional $\mathfrak{L}(\mathfrak{g}, \psi)$ -modules in the Weyl modules. Due to cors. 3.2 and 3.3, it remains to find the block decomposition of $\mathbf{Rep}(\mathfrak{L}(\mathfrak{g}, \psi))$ to complete the classification of finite-dimensional $\mathfrak{L}(\mathfrak{g}, \psi)$ -modules. In complete analogy to thm. 2.23, we define a class of twisted spectral characters to classify indecomposable blocks of $\mathbf{Rep}(\mathfrak{L}(\mathfrak{g}, \psi))$. To this end, we make use of the close relation between twisted and untwisted Drinfeld polynomials.

Def. 3.15. Let $\chi_1, \chi_2 \in \Xi$. Define an equivalence relation $\sim$ on Ξ by requiring that

$$\chi_1 \sim \chi_2 \Leftrightarrow \exists \pi_1, \pi_2 \in \mathcal{P} : \chi_1 = \chi_{\pi_1}, \chi_2 = \chi_{\pi_2} \text{ and } \text{res}(\pi_1) = \text{res}(\pi_2).$$

Def. 3.16. Define the monoid of twisted spectral characters $\tilde{\Xi}$ of $\mathfrak{L}(\mathfrak{g}, \psi)$ to be the partition of the spectral characters Ξ of $\mathfrak{L}(\mathfrak{g})$ into equivalence classes under the equivalence relation of def. 3.15. More concisely, $\tilde{\Xi} = \Xi / \sim$. For $\tilde{\pi} \in \tilde{\mathcal{P}}$, define the twisted spectral character $\chi_{\tilde{\pi}}$ of $\tilde{\pi}$ as $\chi_{\tilde{\pi}} = \chi_{\pi}$ for $\pi \in \text{ind}(\tilde{\pi})$.

Def. 3.17. An element $V \in \text{Obj}(\mathbf{Rep}(\mathfrak{L}(\mathfrak{g}, \psi)))$ is defined to have twisted spectral character $\chi \in \tilde{\Xi}$ if for every irreducible component $V(\tilde{\pi})$ of V it holds that $\chi = \chi_{\tilde{\pi}}$ for the twisted spectral character $\chi_{\tilde{\pi}}$ of the parametrising twisted Drinfeld polynomial $\tilde{\pi} \in \tilde{\mathcal{P}}$. Denote by $\mathbf{C}_{\chi}$ the abelian subcategory of $\mathbf{Rep}(\mathfrak{L}(\mathfrak{g}, \psi))$ that contains all objects with twisted spectral character χ .

Similarly to def. 2.19, the content of def. 3.17 may be formalised by considering the irreducibles in the Jordan-Hölder series of any finite-dimensional $\mathfrak{L}(\mathfrak{g}, \psi)$ -module. Turning, then, to the classification of indecomposable blocks, the following results are analogues to props. 2.21 and 2.22.

Prop. 3.18. *Let $\chi \in \tilde{\Xi}$. Any two irreducible objects in $\mathbf{C}_{\chi}$ are linked.*

Proof. [Senesi 2010, prop. 3.14]. □

Prop. 3.19. *Any indecomposable object in $\mathbf{Rep}(\mathfrak{L}(\mathfrak{g}, \psi))$ admits a twisted spectral character.*

Proof. [Senesi 2010, prop. 3.15]. □

Finally, we give the block decomposition of $\mathbf{Rep}(\mathfrak{L}(\mathfrak{g}, \psi))$, thus completing our categorical classification of finite-dimensional $\mathfrak{L}(\mathfrak{g}, \psi)$ -modules.

Thm. 3.20. *The category $\mathbf{Rep}(\mathfrak{L}(\mathfrak{g}, \psi))$ admits the block decomposition*

$$\mathbf{Rep}(\mathfrak{L}(\mathfrak{g}, \psi)) = \bigoplus_{\chi \in \tilde{\Xi}} \mathbf{C}_{\chi},$$

where each $\mathbf{C}_{\chi}$ is a block. In particular, the blocks of $\mathbf{Rep}(\mathfrak{L}(\mathfrak{g}, \psi))$ are in bijective correspondence with twisted spectral characters $\tilde{\Xi}$ of $\mathfrak{L}(\mathfrak{g}, \psi)$.

Proof. By prop. 3.19, any indecomposable object of $\mathbf{Rep}(\mathfrak{L}(\mathfrak{g}, \psi))$ is in $\mathbf{C}_{\chi}$ for some $\chi \in \tilde{\Xi}$, ensuring the orthogonality of the decomposition into indecomposables. It follows, then, from prop. 3.18 that the $\mathbf{C}_{\chi}$ indeed form blocks. □

4 Representation Theory of General Linear Loop Algebras $\mathfrak{L}(\mathfrak{gl}_n)$

We now slightly generalise the construction of loop algebras $\mathfrak{L}(\mathfrak{g})$ from section 1. Let $n \in \mathbb{N}$. Denote by

$$\mathfrak{gl}_n := \mathfrak{gl}_n(\mathbb{C}) \cong \text{End}(\mathbb{C}^n)$$

the Lie algebra of $(n \times n)$ -indexed matrices $\text{Mat}(n, \mathbb{C})$ with Lie bracket given by the commutator, and define its loop algebra

$$\mathfrak{L}(\mathfrak{gl}_n) := \mathfrak{gl}_n \otimes_{\mathbb{C}} \mathfrak{L}$$

in analogy with def. 1.3. While $\mathfrak{gl}_n$ is a reductive Lie algebra, it is not semisimple. As such, our results regarding central extensions and the representation theory of loop algebras $\mathfrak{L}(\mathfrak{g})$ from sections 1 and 2 do not simply go through unchanged. However, we will be able to take this construction quite far by other means. Define

$$\mathfrak{gl}_{\infty} := \mathfrak{gl}_{\infty}(\mathbb{C}) \cong \text{End}(\mathbb{C}^{\infty})$$

as the space of $(\mathbb{Z} \times \mathbb{Z})$ -indexed matrices $\text{Mat}(\mathbb{N}_0, \mathbb{C})$ of countable dimension. This space is much too large to admit a Lie algebra structure as it does not allow for a generally convergent product. We thus restrict to a much smaller Lie algebra.

Def. 4.1. Define the Lie algebra of finitary matrices as the direct limit Lie algebra

$$\bar{\mathfrak{gl}}_{\infty} := \varinjlim \mathfrak{gl}_n = \{A \in \mathfrak{gl}_{\infty} \mid \text{all but finitely many } A_{ij} - \delta_{ij} \text{ are zero}\},$$

where the Lie bracket is given by the commutator.

Note that convergence of the Lie bracket in $\bar{\mathfrak{gl}}_{\infty}$ is always guaranteed due to finitude of off-diagonal entries. While this Lie algebra is interesting in its own right, it turns out to be much too small for most applications. A motivation for this is, in part, given by the contrast between thms. 4.17 and 4.18. Therefore, we further define a larger Lie algebra.

Def. 4.2. Define the Lie algebra of generalised Jacobian matrices

$$\bar{\mathfrak{a}}_{\infty} = \{A \in \mathfrak{gl}_{\infty} \mid \exists N \in \mathbb{N}: |i - j| \geq N \Rightarrow A_{ij} = 0\},$$

where the Lie bracket is given by the commutator.

Note that $\bar{\mathfrak{gl}}_{\infty} \leq \bar{\mathfrak{a}}_{\infty}$ is a proper subalgebra. The generalised Jacobian matrices $\bar{\mathfrak{a}}_{\infty} \subset \mathfrak{gl}_{\infty}$ may be thought of as the Lie algebra of countably indexed matrices that have only finitely many non-zero diagonals. By this analogy, convergence of

the Lie bracket in $\bar{\mathfrak{a}}_\infty$ is clearly guaranteed. Denote by $E_{ij} \in \bar{\mathfrak{a}}_\infty$ the generalised Jacobian matrix given by unit entry in the $(i, j)^{\text{th}}$ position and zeros everywhere else. Define further the shift operators $\Lambda_k \in \bar{\mathfrak{a}}_\infty$ for $k \in \mathbb{Z}$

$$\Lambda_k = \sum_{i \in \mathbb{Z}} E_{i, i+k},$$

containing units on the k^{th} diagonal and zeros everywhere else. The space spanned by the class of shift operators $\Lambda := \bigoplus_{k \in \mathbb{Z}} \mathbb{C} \Lambda_k \leq \bar{\mathfrak{a}}_\infty$ forms an abelian subalgebra.

Lem. 4.3. *A basis for $\mathfrak{L}(\mathfrak{gl}_n)$ is given by $e_{ij}^{(k)} = e_{ij} \otimes t^k$, where $e_{ij} \in \mathfrak{gl}_n$ is the matrix with unit entry in the ij -th position and zero entries everywhere else, for $1 \leq i, j \leq n$ and $k \in \mathbb{Z}$.*

Proof. Recall that $\{(e_{ij})_{(i,j) \in \mathbb{Z}_n^2}\}$ is a basis for $\mathfrak{gl}_n$ and that $\{(t^k)_{k \in \mathbb{Z}}\}$ is a basis for $\mathfrak{L}$. A basis for a tensored space is given by the tensor combinations of the component space bases. $\square$

We now construct $\mathfrak{L}(\mathfrak{gl}_n)$ as a subalgebra of $\bar{\mathfrak{a}}_\infty$.

Lem. 4.4. *The map $\tau: \mathfrak{L}(\mathfrak{gl}_n) \rightarrow \bar{\mathfrak{a}}_\infty$, defined on the basis elements of $\mathfrak{L}(\mathfrak{gl}_n)$ by*

$$\tau(e_{ij}^{(k)}) = \sum_{s \in \mathbb{Z}} E_{ns+i, n(s+k)+j}$$

and then linearly extending, is a Lie algebra inclusion.

Proof. We want to show that τ is an injective homomorphism of Lie algebras. Since the Lie brackets in $\mathfrak{gl}_n$ and $\bar{\mathfrak{a}}_\infty$ are commutators, it suffices to show that τ is an injective homomorphism of $\mathbb{C}$ -algebras. To this end, suppose $\tau(e_{ij}^{(k)}) = \tau(e_{i'j'}^{(k')})$. By definition, this is equivalent to

$$\sum_{s \in \mathbb{Z}} E_{ns+i, n(s+k)+j} = \sum_{s \in \mathbb{Z}} E_{ns+i', n(s+k')+j'},$$

whence we conclude that $e_{ij}^{(k)} = e_{i'j'}^{(k')}$. $\square$

One may think of the embedding τ from lem. 4.4 as a transformation of $a(t) \in \mathfrak{L}(\mathfrak{gl}_n)$ into a matrix of generalised Toeplitz form in $\bar{\mathfrak{a}}_\infty$, pictorially given as

$$a(t) = \sum_{i \in I} a_i \otimes t^i \mapsto \begin{bmatrix} \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & a_0 & a_1 & \ddots & \ddots \\ \ddots & a_{-1} & a_0 & a_1 & \ddots \\ \ddots & \ddots & a_{-1} & a_0 & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix},$$

where $a_i \in \mathfrak{gl}_n$ and I is a finite index set, ensuring the existence of only finitely many non-zero diagonals. Since each diagonal is occupied by identical elements $a_i \in \mathfrak{gl}_n$, $\tau(a(t))$ is an n -periodic element of $\bar{\mathfrak{a}}_\infty$.

By lem. 4.4, $\mathfrak{V}(\mathfrak{gl}_n) \leq \bar{\mathfrak{a}}_\infty$ is a proper subalgebra. As such, any representation of $\bar{\mathfrak{a}}_\infty$ canonically restricts to a representation of $\mathfrak{V}(\mathfrak{gl}_n)$. Thus, we turn to the study of $\bar{\mathfrak{a}}_\infty$ -modules. The most interesting of these occur in an infinite-dimensional vector space called semi-infinite wedge space.

Def. 4.5. Let $V = \bigoplus_{i \in \mathbb{Z}} \mathbb{C}v_i$ be an infinite-dimensional, complex vector space with a fixed basis given by the collection of v_i . Define the semi-infinite wedge space

$$\bigwedge^{\frac{\infty}{2}} V = \bigoplus_{m \in \mathbb{Z}} \bigwedge^{\frac{\infty}{2}, m} V,$$

where

$$\bigwedge^{\frac{\infty}{2}, m} V = \text{span}_{\mathbb{C}} \{ \psi = v_{i_m} \wedge v_{i_{m-1}} \wedge v_{i_{m-2}} \wedge \dots \mid i_m > i_{m-1} > \dots \text{ and } i_k = k \text{ for } k \ll 0 \}$$

is the linear span of certain semi-infinite monomials. We define a positive, Hermitian form $\langle \cdot, \cdot \rangle$ on $\bigwedge^{\frac{\infty}{2}} V$ by declaring the class of semi-infinite monomials to be an orthonormal basis with respect to $\langle \cdot, \cdot \rangle$. Furthermore, on each subspace of this grading, we may define the m^{th} vacuum vector

$$\psi_m = \bigwedge_{k=-m}^{\infty} v_{-k} = v_m \wedge v_{m-1} \wedge v_{m-2} \wedge \dots \in \bigwedge^{\frac{\infty}{2}, m} V.$$

Define further the perfect vacuum vector

$$\psi_0 = \bigwedge_{k=0}^{\infty} v_{-k} = v_0 \wedge v_{-1} \wedge v_{-2} \wedge \dots \in \bigwedge^{\frac{\infty}{2}, 0} V$$

as the zeroth vacuum vector.

We define the degree of any $\psi \in \bigwedge^{\frac{\infty}{2}, m} V$ as

$$\deg(\psi) = \sum_{j=0}^{\infty} (i_{m-j} + j - m).$$

Since for $k \ll 0 : i_k = k$, any such degree is a finite positive number. This naturally leads, for any $m \in \mathbb{Z}$, to a grading

$$\bigwedge^{\frac{\infty}{2}, m} V = \bigoplus_{k \geq 0} \bigwedge^{\frac{\infty}{2}, m}_{(k)} V$$

of subspaces of fixed degrees k . The degree zero subspace herein is

$$\bigwedge_{(0)}^{\frac{\infty}{2}, m} V = \mathbb{C}\psi_m,$$

precisely the one spanned by the vacuum vector

Prop. 4.6. *Let $m \in \mathbb{Z}$ and $k \geq 0$. Then*

$$\dim_{\mathbb{C}}(\bigwedge_{(k)}^{\frac{\infty}{2}, m} V) = p(k),$$

where, for $k \in \mathbb{N}$, $p(k)$ denotes the number of partitions of k .

Proof. The degree of any $\psi \in \bigwedge^{\frac{\infty}{2}, m} V$ is clearly a finite, nonnegative integer due to the condition that $i_j = j$ for $j \ll 0$ and the requirement that the i_j be strictly monotonically decreasing in j from def. 4.5. Let, therefore, $k \in \mathbb{N}$. Any partition $(k_0, k_1, \dots, k_{n-1})$ of k in non-increasing order defines a unique $\psi \in \bigwedge_{(k)}^{\frac{\infty}{2}, m}$ by setting

$$\psi = \bigwedge_{i=0}^{\infty} v_{k_i - i},$$

where it is understood that $k_j = 0$ for $j \geq n$. Since this assignment between partitions of k and semi-infinite wedge vectors ψ of degree k is bijective, we conclude the claim. $\square$

Cor. 4.7. *Let $m \in \mathbb{Z}$. Then, as a formal sum, it holds that*

$$\dim_q(\bigwedge^{\frac{\infty}{2}, m} V) := \sum_{k \in \mathbb{N}} (\dim_{\mathbb{C}}(\bigwedge_{(k)}^{\frac{\infty}{2}, m} V)) q^k = \prod_{k \in \mathbb{N}} (1 - q^k)^{-1} = \varphi(q)^{-1},$$

where φ is Euler's function. We call this the q -dimension of $\bigwedge^{\frac{\infty}{2}, m} V$.

Proof. This follows directly from prop. 4.6 and a famous identity due to Euler, whence

$$\sum_{k \in \mathbb{N}} (\dim_{\mathbb{C}}(\bigwedge_{(k)}^{\frac{\infty}{2}, m} V)) q^k = \sum_{k \in \mathbb{N}} p(k) q^k = \frac{1}{\prod_{k \in \mathbb{N}} (1 - q^k)}.$$

$\square$

Rem. 4.8. For each $m \in \mathbb{Z}$, the semi-infinite wedge space $\bigwedge^{\frac{\infty}{2}, m} V$ is, in principle, nothing more than the fermionic Fock space $F^{(m)}$ with a vacuum state at charge number m . We may, therefore, think of the collection $\bigwedge^{\frac{\infty}{2}} V$ of these semi-infinite wedge spaces as a total fermionic Fock space, incorporating in it all possible charge numbers.

We now define the infinite wedge representation of $\bar{\mathfrak{gl}}_\infty$ in the semi-infinite wedge space. Subsequently, we enlarge this to a projective representation of $\bar{\mathfrak{a}}_\infty$. Define the $\bar{\mathfrak{gl}}_\infty$ -module structure on $\bigwedge^{\frac{\infty}{2}} V$ by requiring that, as a Lie algebra, $\bar{\mathfrak{gl}}_\infty$ act by derivation

$$\varrho : \bar{\mathfrak{gl}}_\infty \rightarrow \text{End}(\bigwedge^{\frac{\infty}{2}} V); a \mapsto \sum_{j=1}^{\infty} \text{id}^{\wedge(j-1)} \wedge a \wedge \text{id}^{\wedge \mathbb{N}_0},$$

where $a \in \bar{\mathfrak{gl}}_\infty$ acts on $v \in V$ by left multiplication. Explicitly, on the basis elements $E_{ij} \in \bar{\mathfrak{gl}}_\infty$, this representation takes the form

$$\varrho(E_{ij})(v_{i_1} \wedge v_{i_2} \wedge \dots) = \begin{cases} v_{i_1} \wedge \dots \wedge v_{i_{k-1}} \wedge v_i \wedge v_{i_{k+1}} \wedge \dots, & \text{if } j = i_k, \\ 0, & \text{if } \forall k : j \neq i_k. \end{cases} \quad (4.1)$$

By this representation, we may rewrite any element $\psi \in \bigwedge^{\frac{\infty}{2}, m} V$ as

$$\psi = \varrho(E_{i_m, m}) \dots \varrho(E_{i_{m-k}, m-k}) \psi_m \quad (4.2)$$

for a finite number of generators $E_{ij, j} \in \bar{\mathfrak{gl}}_\infty$. From this structure, we find the following claim.

Prop. 4.9. *The $\bar{\mathfrak{gl}}_\infty$ -representation ϱ in $\bigwedge^{\frac{\infty}{2}} V$ is a direct sum*

$$\varrho = \bigoplus_{m \in \mathbb{Z}} \varrho_m : \bar{\mathfrak{gl}}_\infty \rightarrow \text{End}(\bigoplus_{m \in \mathbb{Z}} \bigwedge^{\frac{\infty}{2}, m} V)$$

of irreducible, unitary $\bar{\mathfrak{gl}}_\infty$ -representations ϱ_m in $\bigwedge^{\frac{\infty}{2}, m} V$.

Proof. The direct sum decomposition follows directly from the orthogonality of the decomposition in def. 4.5 by noting that the $\bar{\mathfrak{gl}}_\infty$ -action through ϱ maps each $\bigwedge^{\frac{\infty}{2}, m} V$ into itself, which is most clearly seen in the explicit form of equation (4.1) on the generators. Unitarity follows from

$$\langle \varrho(E_{ij})\psi_1, \psi_2 \rangle = \langle \psi_1, \varrho(E_{ij}^\dagger)\psi_2 \rangle,$$

for $\psi_1, \psi_2 \in \bigwedge^{\frac{\infty}{2}, m} V$, after linear extension. It remains to show irreducibility. Let $\bigwedge^{\frac{\infty}{2}, m} V = V_1 \oplus V_2$ be the direct sum decomposition of two ϱ_m -subrepresentations. Either of the subrepresentations then naturally inherits the $\mathbb{N}$ -grading of $\bigwedge^{\frac{\infty}{2}, m} V$, namely

$$V_i = \bigoplus_{k \geq 0} (\bigwedge_{(k)}^{\frac{\infty}{2}, m} V \cap V_i).$$

The vacuum vector ψ_m spans the space of degree zero. Therefore, it belongs to exactly one summand, whence, by equation (4.2), all of $\bigwedge^{\frac{\infty}{2}, m} V$ must be within that summand. We conclude that the other summand must be 0 and that $\bigwedge^{\frac{\infty}{2}, m} V$ is irreducible. $\square$

On the $\mathbb{N}$ -gradings $V_i = \bigoplus_{k \geq 0} \bigwedge_{(k)}^{\frac{\infty}{2}, m} V$, the $\bar{\mathfrak{gl}}_\infty$ -action of ϱ takes, by equation (4.2), the form

$$\varrho(E_{ij}) \bigwedge_{(k)}^{\frac{\infty}{2}, m} V \subset \bigwedge_{(k+i-j)}^{\frac{\infty}{2}, m} V.$$

We may use this to naturally induce a $\mathbb{Z}$ -grading on $\bar{\mathfrak{gl}}_\infty$.

Def. 4.10. Define the degree of the generators $E_{ij} \in \bar{\mathfrak{gl}}_\infty$ to be $\deg(E_{ij}) = i - j$. Using these, define the principal gradation of $\bar{\mathfrak{gl}}_\infty$ as the decomposition

$$\bar{\mathfrak{gl}}_\infty = \bigoplus_{j \in \mathbb{Z}} \bar{\mathfrak{gl}}_\infty^{(j)},$$

into spaces $\bar{\mathfrak{gl}}_\infty^{(j)}$ of homogeneous degree j .

The principal gradation of $\bar{\mathfrak{gl}}_\infty$ is clearly well-defined, since the degrees are, by construction, additive. Furthermore, the construction turns $\bigwedge_{(k)}^{\frac{\infty}{2}, m} V$ into a $\mathbb{Z}$ -graded $\bar{\mathfrak{gl}}_\infty$ -module, since for any $j, m \in \mathbb{Z}$ and any $k \in \mathbb{N}$, it holds that

$$\varrho(\bar{\mathfrak{gl}}_\infty^{(j)}) \bigwedge_{(k)}^{\frac{\infty}{2}, m} V \subset \bigwedge_{(k+j)}^{\frac{\infty}{2}, m} V.$$

Rem. 4.11. Using the principal gradation from def. 4.10 and equation (4.2), we may rewrite the k^{th} summand in the orthogonal decomposition of the semi-infinite wedge space as

$$\bigwedge_{(k)}^{\frac{\infty}{2}, m} V = \sum_{\bar{k} \in \text{Part}(k)} \varrho_m(\bar{\mathfrak{gl}}_\infty^{(k_0)}) \dots \varrho_m(\bar{\mathfrak{gl}}_\infty^{(k_{n-1})}) \psi_m,$$

where $\bar{k} = (k_0, \dots, k_{n-1})$ denotes an integer partition of k . This gives a representation-theoretic interpretation of the fermionic Fock space with charge number m . In particular, it sheds representation-theoretic light on the Dirac positron model, wherein deviations from the vacuum vector ψ_m with index greater than m denote electrons and deviations from indices smaller than m denote holes. These holes are, then, interpreted as positrons. This explicit form gives a representation-theoretic construction for any arrangement of electrons and positrons from a vacuum state. In this interpretation, the degree of a semi-infinite vector corresponds to the energy of the system.

By the principal gradation of $\bar{\mathfrak{gl}}_\infty$ from def. 4.10, we may define a triangular decomposition.

Def. 4.12. Consider the principal gradation of $\bar{\mathfrak{gl}}_\infty$ from def. 4.10. Define the triangular decomposition of $\bar{\mathfrak{gl}}_\infty$ as

$$\bar{\mathfrak{gl}}_\infty = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-,$$

where

$$\mathfrak{n}_+ = \bigoplus_{j < 0} \bar{\mathfrak{gl}}_\infty^{(j)}, \quad \mathfrak{n}_- = \bigoplus_{j > 0} \bar{\mathfrak{gl}}_\infty^{(j)},$$

and where $\mathfrak{h} = \bar{\mathfrak{gl}}_\infty^{(0)}$ is called the Cartan subalgebra of $\bar{\mathfrak{gl}}_\infty$.

As is usual with Lie algebras that admit a triangular decomposition, a natural class of modules to consider is the class of highest weight modules.

Def. 4.13. Let $\lambda = \{\lambda_i \in \mathbb{C} \mid i \in \mathbb{Z}\}$ be a collection of numbers. An irreducible highest weight representation π_λ of $\bar{\mathfrak{gl}}_\infty$ is an irreducible representation

$$\pi_\lambda : \bar{\mathfrak{gl}}_\infty \rightarrow \text{End}(L(\lambda))$$

on a vector space $L(\lambda)$ that admits a vector $v_\lambda \in L(\lambda)$, called highest weight vector, with $\pi_\lambda(\mathfrak{U}(\bar{\mathfrak{gl}}_\infty))v_\lambda = L(\lambda)$ such that

$$\pi_\lambda(\mathfrak{n}_+)v_\lambda = 0 \quad \text{and} \quad \forall i \in \mathbb{Z} : \pi_\lambda(E_{ii})v_\lambda = \lambda_i v_\lambda.$$

We call λ the corresponding highest weight.

Alternatively, we may, instead of the collection λ , define a linear functional $\lambda \in \mathfrak{h}^*$ by requiring it precisely to yield $\lambda(E_{ii}) = \lambda_i$. The restatement then follows the definition of classical highest weight modules. The following clarifies the uniqueness of highest weight representations for any given highest weight.

Prop. 4.14. For any $\lambda = \{\lambda_i \in \mathbb{C} \mid i \in \mathbb{Z}\}$, there exists a unique highest weight $\bar{\mathfrak{gl}}_\infty$ -module $L(\lambda)$.

Proof. The proof goes through in the typical fashion for highest weight representations. Namely, we construct the representation explicitly through a Verma module, whereby we show its uniqueness. To this end, let $\mathbb{C}_\lambda$ be the 1-dimensional $\mathfrak{U}(\mathfrak{h} \oplus \mathfrak{n}_+)$ -module spanned by a vector v , on which $\mathfrak{h}$ acts by the linear functional $\lambda \in \mathfrak{h}^*$ defined by the collection λ , and the $\mathfrak{n}_+$ -action is trivial. Define, then, the Verma module $M(\lambda)$ to the highest weight λ

$$M(\lambda) = \mathfrak{U}(\bar{\mathfrak{gl}}_\infty) \otimes_{\mathfrak{U}(\mathfrak{h} \oplus \mathfrak{n}_+)} \mathbb{C}_\lambda.$$

By the structure of the universal enveloping algebra, $\mathfrak{U}(\bar{\mathfrak{gl}}_\infty) \cong \mathfrak{U}(\mathfrak{n}_-) \otimes \mathfrak{U}(\mathfrak{h} \oplus \mathfrak{n}_+)$, whence $M(\lambda)$ is a free $\mathfrak{U}(\mathfrak{n}_-)$ -module with a single generator v_λ . Since $\mathfrak{U}(\mathfrak{n}_-)$ is by definition $\mathbb{N}$ -graded, any weight vector in $M(\lambda)$ is of the form uv_λ with weight $\lambda + \alpha$ for $u \in \mathfrak{U}(\mathfrak{n}_-)_{\alpha}$. As such, the $\mathbb{N}$ -grading on $\mathfrak{U}(\mathfrak{n}_-)$ induces an $\mathbb{N}$ -grading on $M(\lambda)$. Then, by the same argument as in the proof of prop. 4.9, $M(\lambda)$ is indecomposable. Furthermore, $M(\lambda)$ admits a unique, maximal, graded submodule $I(\lambda)$. Indeed,

taking $I(\lambda)$ to be the sum of all submodules suffices, since any such submodule must also be graded and cannot contain v_λ . Taking the quotient

$$L(\lambda) = M(\lambda)/I(\lambda)$$

defines the unique highest weight $\bar{\mathfrak{gl}}_\infty$ -module of highest weight λ with highest weight vector v_λ . $\square$

Prop. 4.15. *For each $m \in \mathbb{Z}$, the irreducible $\bar{\mathfrak{gl}}_\infty$ -representation ϱ_m in the decomposition of the infinite wedge representation ϱ from prop. 4.9 is a highest weight representation with highest weight $\omega_m = \{\forall i \leq m : \lambda_i = 1, \text{ and } \forall i > m : \lambda_i = 0\}$.*

Proof. From equation (4.2), we find that, taking ψ_m as the highest weight vector of $\wedge^{\frac{\infty}{2}, m} V$, we may generate all of $\wedge^{\frac{\infty}{2}, m} V$ by the $\bar{\mathfrak{gl}}_\infty$ -action via ϱ_m . By equation (4.1), we find that $\varrho_m(\mathfrak{n}_+)(\psi_m) = 0$, since $\mathfrak{n}_+$ contains only elements E_{ij} with $i < j$, leading, by necessity, to double occurrences of factors in $\varrho_m(E_{ij})(\psi_m)$, and thus, by alternativity of the wedge product, equating to zero. It remains to show that $\varrho_m(E_{ii})\psi_m = \lambda_i\psi_m$ for $\lambda_i \in \mathbb{C}$. However, defining the λ_i via the collection ω_m as above, the claim clearly follows. $\square$

We call the irreducible unitary $\bar{\mathfrak{gl}}_\infty$ -representations ϱ_m the fundamental representations of $\bar{\mathfrak{gl}}_\infty$ with fundamental weights ω_m . It follows by construction from prop. 4.9 that the infinite wedge representation ϱ is a direct sum of fundamental representations. Using the unitarity of the fundamental representations, we may directly deduce unitarity for a large class of highest weight $\bar{\mathfrak{gl}}_\infty$ -representations.

Prop. 4.16. *A highest weight $\bar{\mathfrak{gl}}_\infty$ -representation with highest weight $\sum_{i \in \mathbb{Z}} k_i \omega_i$, where $k_i \in \mathbb{N}$ are nonnegative, is unitary.*

Proof. The highest weight representation of highest weight $\sum_{i \in \mathbb{Z}} k_i \omega_i$ may be explicitly constructed as the tensor product of fundamental $\bar{\mathfrak{gl}}_\infty$ -modules

$$\bigotimes_{i \in \mathbb{Z}} (\varrho_i)^{\otimes k_i},$$

since highest weights are additive under products of highest weight representations. By prop. 4.14, this is, up to isomorphism, the unique highest weight representation of highest weight $\sum_{i \in \mathbb{Z}} k_i \omega_i$. Since the tensor product of two unitary representations is, again, unitary, and since, by prop. 4.9, each ϱ_i is unitary, we conclude the claim. $\square$

We could try to extend the $\bar{\mathfrak{gl}}_\infty$ -representation to a representation of $\bar{\mathfrak{a}}_\infty$ by the induced module. However, this approach quickly runs into issues with convergence, and is, therefore, an invalid extension. To illustrate, we take an element

$a = \sum_{i \in \mathbb{Z}} \lambda_i E_{ii} \in \bar{\mathfrak{a}}_\infty$ across the principal diagonal for arbitrary scalars $\lambda_i \in \mathbb{C}$. Denoting, by abuse of notation, the extended representation by ϱ , we find

$$\varrho(a)\psi_m = \sum_{i=-m}^{\infty} \lambda_{-i} \psi_m,$$

which, as a countably infinite complex series, is, in general, not convergent. This problem may be remedied by considering the central extension of $\bar{\mathfrak{a}}_\infty$, giving an additional degree of freedom to rectify the convergence issue. However, we first formalise the notion of smallness for $\bar{\mathfrak{gl}}_\infty$ in thm. 4.17.

Thm. 4.17. *The second algebraic cohomology of the finitary matrices $\bar{\mathfrak{gl}}_\infty$, $H_{alg}^2(\bar{\mathfrak{gl}}_\infty; \mathbb{C})$, is trivial*

$$H_{alg}^2(\bar{\mathfrak{gl}}_\infty; \mathbb{C}) = 0.$$

Proof. [Verdier 1982]. □

We now turn to the central extensions of $\bar{\mathfrak{a}}_\infty$.

Thm. 4.18. *The second algebraic cohomology of the generalised Jacobian matrices, $H_{alg}^2(\bar{\mathfrak{a}}_\infty; \mathbb{C})$, is 1-dimensional. It is spanned by the 2-cocycle ζ , which, for $a, b \in \bar{\mathfrak{a}}_\infty$, takes the form*

$$\zeta(a, b) = \frac{1}{2} \text{tr}_{\mathbb{C}}([J, a]b),$$

where $J \in \bar{\mathfrak{a}}_\infty$ is the generalised symplectic matrix, given by

$$J = \sum_{j \in \mathbb{Z}} \epsilon(j) E_{jj},$$

where

$$\epsilon(j) = \begin{cases} -1, & \text{if } j \geq 0, \\ +1, & \text{if } j < 0. \end{cases}$$

Proof. [Verdier 1982, 2.7]. □

By the general classification theorem of central extensions by second cohomologies, we conclude the following result.

Cor. 4.19. *There exists, up to isomorphism, a unique, 1-dimensional central extension*

$$\mathfrak{a}_\infty = \bar{\mathfrak{a}}_\infty \oplus \mathbb{C}K, \tag{4.3}$$

where K is a central element. The Lie bracket on $\mathfrak{a}_\infty$ is induced by the one on $\bar{\mathfrak{a}}_\infty$ and the 2-cocycle ζ from thm. 4.18.

Lem. 4.20. *The 2-cocycle ζ in the central extension $\mathfrak{a}_\infty = \bar{\mathfrak{a}}_\infty \oplus \mathbb{C}K$ is given on the basis elements $E_{ij} \in \bar{\mathfrak{a}}_\infty$ by*

$$\begin{aligned}\zeta(E_{ij}, E_{ji}) &= 1, \quad \text{if } i \leq 0, j \geq 1, \\ \zeta(E_{ij}, E_{mn}) &= 0, \quad \text{otherwise.}\end{aligned}$$

Proof. Note that, in more explicit form, the 2-cocycle ζ spanning $H_{alg}^2(\bar{\mathfrak{a}}_\infty; \mathbb{C})$ from thm. 4.18 may, for $a, b \in \bar{\mathfrak{a}}_\infty$, be written as

$$\zeta(a, b) = \frac{1}{2} \sum_{(i,j) \in \mathbb{Z}^2} (\epsilon(i) - \epsilon(j)) a_{ij} b_{ji}.$$

Inserting combinations of generators E_{ij}, E_{mn} , the claims follow. $\square$

These results now allow us to extend the $\bar{\mathfrak{gl}}_\infty$ -representation ϱ to a linear representation $\hat{\varrho}$ of $\mathfrak{a}_\infty$. Indeed, defining the action of $\hat{\varrho}$ by the unitary irreducibles to be

$$\begin{aligned}\hat{\varrho}_m(E_{ij}) &= \varrho_m(E_{ij}), \quad \text{if } i \neq j \text{ or } i = j > 0, \\ \hat{\varrho}_m(E_{ii}) &= \varrho_m(E_{ii}) - \text{id}, \quad \text{if } i \leq 0, \\ \hat{\varrho}_m(K) &= \text{id}.\end{aligned}$$

Due to the commutation relations in $\bar{\mathfrak{a}}_\infty$, this was previously not possible. However, the commutation relations of $\mathfrak{a}_\infty$, induced by those on $\bar{\mathfrak{a}}_\infty$ and the 2-cocycle ζ , precisely give the above structure. As $\hat{\varrho}$ is a linear representation of $\mathfrak{a}_\infty$, its restriction to $\bar{\mathfrak{a}}_\infty$ defines, therefore, a projective representation of $\bar{\mathfrak{a}}_\infty$. Furthermore, naturally extending the Hermitian structure of $\bar{\mathfrak{gl}}_\infty$ to $\mathfrak{a}_\infty$ via $\omega(c) = c$, $\hat{\varrho}_m$ is again unitary. By the same construction, any unitary, irreducible, highest weight $\bar{\mathfrak{gl}}_\infty$ -module extends to a unitary $\mathfrak{a}_\infty$ -module, and, therefore, to a projective representation of $\bar{\mathfrak{a}}_\infty$. By the inclusion from lem. 4.4, this gives a large class of unitary, irreducible, highest weight $\mathfrak{L}(\mathfrak{gl}_n)$ -modules.

On the centrally extended subalgebra $\Lambda \oplus \mathbb{C}K$ spanned by the shift operators Λ_k and the central element K , the extended representation takes a particularly simple form. The following is an implication of lem. 4.20.

Cor. 4.21. *On the subalgebra $\Lambda \leq \bar{\mathfrak{a}}_\infty$ spanned by the shift operators $\Lambda_k \in \bar{\mathfrak{a}}_\infty$, the 2-cocycle ζ from the central extension $\mathfrak{a}_\infty = \bar{\mathfrak{a}}_\infty \oplus \mathbb{C}K$ takes the form*

$$\zeta(\Lambda_n, \Lambda_m) = n\delta_{n,-m}.$$

Therefore, since $\Lambda \leq \bar{a}_\infty$ is abelian, the commutation relations under $\hat{\varrho}_m$ take the following simple form

$$\begin{aligned} [\hat{\varrho}_m(\Lambda_n), \hat{\varrho}_m(\Lambda_k)] &= \hat{\varrho}_m([\Lambda_n + \Lambda_k]) + \zeta(\Lambda_n, \Lambda_k) \hat{\varrho}_m(K) \\ &= n\delta_{n, -k}. \end{aligned}$$

Rem. 4.22. These commutation relations show that Λ defines a representation of the Heisenberg algebra. In fact, precisely this structure is used to deduce the Boson-Fermion correspondence – a phenomenon that occurs in $(1+1)$ -dimensional conformal field theories and defines a correspondence between representations on the bosonic Fock space and on the fermionic Fock space. In particular, it may be used to deduce highly non-trivial functional identities.

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