
Canonical scattering amplitudes on the commutative graph complex

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1 Preamble

Es ist verständlich, daß eine algebraische Zahlentheorie allein noch nicht den Verdacht erweckt, für die Physik bedeutsam zu sein; denn die mathematische Physik denkt in Funktionen und nicht in Zahlen. Nachdem aber in der bereits vollständig entwickelten Theorie der algebraischen Funktionen einer Variablen über einem Galoisfeld das zahlentheoretische Denken in die Analysis eingedrungen ist, kann man nur noch mit Spannung der völligen Synthese arithmetischen, analytischen und geometrischen Denkens und dem Einsatz eines so geeinten Denkens entgegenstehen.

The above epigraph, recited from [Käh58], describes precisely the situation we find ourselves in more than half a century later. For, the program that we aim to outline sits precisely at an intersection of arithmetic, analytic, and geometric thought. The main object of study – scattering amplitudes – is naturally of analytic nature, while the careful analysis of the required differential forms and their integration domains require a sophisticated algebro-geometric language. Furthermore, much of the explored program is close to – and very often dealt with using methods from – algebraic number theory. We believe that it is this synthesis of approaches and languages that renders the field of canonical scattering amplitudes so fruitful.

This essay lays no claim to originality. The outlined program is taken directly from [Bro21] who pioneered the study of these canonical scattering amplitudes on the graph complex. Frequently, use is made of known results throughout the established literature. In all such cases, care was taken to give reference to the relevant sources.

In addition to a formative mathematical education, we assume that the reader be familiar with algebraic geometry to the extent of Kähler differentials and iterative blow-ups of linear subspaces. Furthermore, we assume that the reader has some familiarity with concepts from homology theory, differential forms, and Lie algebras.

Lastly, but most certainly not least, I would like to extend my sincerest gratitude to Prof. Thomas Willwacher for guiding me throughout this project. I thank him for answering the many questions I had while acquainting myself with this field, and for suggesting to me this topic, thereby introducing me to the deep and exciting themes surrounding the area of canonical scattering amplitudes.

2 A graph complex

Let $G = (V_G, E_G)$ be a connected graph. We denote by h_G the genus of G , i.e. its loop number. Denoting $e_G := |E_G|$, we call $\deg(G) := e_G - 2h_G$ the degree of G . Note that, in this convention, the degree of a graph corresponds to the negative of the 'superficial degree of divergence' in the language of Feynman amplitudes [Wei13]. Oriented graphs carry additional structure in the form of an orientation

$$\eta \in (\det(\mathbb{Z}^{E_G}))^\times, \quad (2.1)$$

which, for $n = e_G$ may be thought of as an ordering $\eta = e_1 \wedge \dots \wedge e_n$ of edges up to the natural action of the alternating group A_n . For ordered graphs, the boundary map ∂ is presumed to respect the ordering. Following Kontsevich [Kon93], we introduce a natural vector space, generated by oriented graphs.

Definition 2.1. Define the graph complex $\mathbb{GC}_2$ to be the $\mathbb{Q}$ -vector space generated by the class of oriented graphs (G, η) without tadpoles or vertices of degree ≤ 2 , modulo the relations:

1. $(G, -\eta) = -(G, \eta)$, and
2. for $\sigma \in \text{Hom}(G, G')$ an isomorphism, $(G, \eta) = (G', \sigma(\eta))$.

We denote equivalence classes in $\mathbb{GC}_2$ by $[G, \eta]$. It turns out that the graph complex $\mathbb{GC}_2$ admits a very rich structure. Define, for any subgraph $\gamma \subset G$, the operation $G \parallel \gamma$ to be equal to the usual contraction G/γ if $h_\gamma = 0$, and equal to $\emptyset$ otherwise. We define a differential $d : \mathbb{GC}_2 \rightarrow \mathbb{GC}_2$ by the requirement

$$d[G, e_1 \wedge \dots \wedge e_n] = \sum_{j=1}^n (-1)^j [G \parallel e_j, e_1 \wedge \dots \wedge \hat{e}_j \wedge \dots \wedge e_n], \quad (2.2)$$

where $\hat{e}_j$ denotes omission of the variable e_j from the Grassmann product.

Lemma 2.2. *The graph differential $d : \mathbb{GC}_2 \rightarrow \mathbb{GC}_2$ is well-defined and nilpotent in second degree.*

Proof. We compute

$$\begin{aligned} d^2[G, e_1 \wedge \dots \wedge e_n] &= d \left(\sum_{j=1}^n (-1)^j [G \parallel e_j, e_1 \wedge \dots \wedge \hat{e}_j \wedge \dots \wedge e_n] \right) \\ &= \sum_{j=1}^n (-1)^j \sum_{i \neq j} (-1)^i [G \parallel \{e_j, e_i\}, e_1 \wedge \dots \wedge \hat{e}_i, \hat{e}_j \wedge \dots \wedge e_n] \end{aligned}$$

$$\begin{aligned}
&= \sum_{1 \leq i < j \leq n} (-1)^{i+j} \left([G \parallel \{e_i, e_j\}, e_1 \wedge \dots \wedge \hat{e}_i, \hat{e}_j \wedge \dots \wedge e_n] \right. \\
&\quad \left. - [G \parallel \{e_i, e_j\}, e_1 \wedge \dots \wedge \hat{e}_j, \hat{e}_i \wedge \dots \wedge e_n] \right) \\
&= 0,
\end{aligned}$$

where the negative sign in the penultimate step arises due to a reversal of orientation with respect to the previous term. The claim follows. $\square$

The fact that the graph differential $d : \mathbb{GC}_2 \rightarrow \mathbb{GC}_2$ is nilpotent in second degree while lowering degrees by one gives us a well-defined differential (Chevalley-Eilenberg) chain complex $(\mathbb{GC}_2, d)$ on the graph complex $\mathbb{GC}_2$. As such, it is natural to define a homology theory on $\mathbb{GC}_2$.

Definition 2.3. Define the Kontsevich (or graph) homology

$$H_\bullet(\mathbb{GC}_2) = \frac{\ker d}{\text{im } d} \quad (2.3)$$

as the homology of the differential chain complex on graphs.

The homology on the graph complex admits a bigrading

$$H_\bullet(\mathbb{GC}_2) = \bigoplus_{n \in \mathbb{Z}} H_n(\mathbb{GC}_2) = \bigoplus_{n \in \mathbb{Z}} \bigoplus_{h \in \mathbb{Z}_{\geq 0}} H_n(\mathbb{GC}_2)^{(h)}, \quad (2.4)$$

where the homological degree n enumerates the degree $\deg(G)$ of the representative graphs $[G, \eta]$ and h enumerates their loop numbers.

Denote by $H_1(G; \mathbb{Z})$ the first homology group on the classical graph homology of cycles. More formally, it is the first homology group of the simplicial homology for the graph simplicial complex (G, ∂) , where the boundary operator ∂ takes edges to formal differences of their incident vertices, respecting orientation in that it subtracts sources from targets. For a given oriented graph $[G, \eta]$, we may construct a short exact sequence

$$0 \longrightarrow H_1(G; \mathbb{Z}) \xrightarrow{\mathcal{H}_G} \mathbb{Z}^{E_G} \xrightarrow{\partial} \mathbb{Z}^{V_G} \xrightarrow{\Sigma} \mathbb{Z} \longrightarrow 0, \quad (2.5)$$

where $\mathcal{H}_G$ maps a 1-cycle in $H_1(G; \mathbb{Z})$ to its incidence on edges. Since the boundary operator ∂ subsequently respects orientation, and since $\mathcal{H}_G$ maps cycles, its image lies in the kernel of ∂ . The map $\Sigma : \mathbb{Z}^{V_G} \rightarrow \mathbb{Z}$ sums coefficients. Since ∂ maps edges to equally weighted and oppositely signed bivariate formal sums, its image is in the kernel of Σ . After assigning to every edge $e \in E_G$ a variable α_e , we may define a symmetric bilinear form

$$\begin{aligned}
\langle \cdot, \cdot \rangle : \mathbb{Z}^{E_G} \times \mathbb{Z}^{E_G} &\longrightarrow \mathbb{Z}[(\alpha_e)_{e \in E_G}]; \\
(e, e') &\longmapsto \delta_{e, e'} \alpha_e.
\end{aligned} \quad (2.6)$$

Note that in the physics literature, the variables $(\alpha_e)_{e \in E_G}$ are known as Schwinger (or, as a special subcase, Feynman) parameters.

Definition 2.4. Let

$$\mathcal{A}_G : \mathbb{Z}^{E_G} \longrightarrow \text{Hom}(\mathbb{Z}^{E_G}, \mathbb{Z}[(\alpha_e)_{e \in E_G}]) \quad (2.7)$$

be the map that sends every edge $e \in E_G$ to the function $\alpha_e e^\vee$ on the dual basis, extending linearly. Define the graph Laplacian

$$\Lambda_G = \mathcal{H}_G^T \mathcal{A}_G \mathcal{H}_G : H_1(G; \mathbb{Z}) \longrightarrow \text{Hom}(H_1(G; \mathbb{Z}), \mathbb{Z}[(\alpha_e)_{e \in E_G}]). \quad (2.8)$$

We define a polynomial graph invariant whose history goes back to the study [Kir47] of galvanic currents in electrical circuits. A century later, it found renewed relevance during investigations [Sym58] into perturbative quantum field theory.

Definition 2.5. Define the Kirchhoff (or 1st Symanzik) polynomial

$$\Psi_G = \sum_{T \in \mathcal{T}(G)} \prod_{e \in (E_G \setminus E_T)} \alpha_e \in \mathbb{Z}[(\alpha_e)_{e \in E_G}]. \quad (2.9)$$

Note that, by construction, $\Psi_G = \det \Lambda_G$. This fact is sometimes referred to as a dual matrix-tree theorem. It is this representation from which we may immediately deduce that Ψ_G is a graph invariant.

Lemma 2.6. *The Kirchhoff polynomial Ψ_G is a well-defined graph invariant.*

Proof. Since $\mathcal{A}_G$ is diagonal in the basis of E_G , it is symmetric. Therefore, Λ_G is symmetric, too. Any other representation of Λ_G by a symmetric matrix corresponds to a change of basis over the integers, which changes the determinant at most by an element of $(\mathbb{Z}^\times)^2$. Since $\mathbb{Z}^\times = \{-1, 1\}$, it follows that $(\mathbb{Z}^\times)^2 = 1$. $\square$

3 Invariant forms

3.1. Extrinsic Maurer-Cartan forms. In this section, we consider, quite generally, a graded-commutative and graded-differential $\mathbb{Q}$ -algebra $R = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} R^n$ with degree 1 differential $d : R^n \rightarrow R^{n+1}$. For $a \in R^n$, we call $\deg(a) = n$ the degree of a . Then, graded-commutative denotes the fact that for any $a, b \in R$ it holds that $ab = (-1)^{\deg(a)\deg(b)}ba$.

Definition 3.1. Let $X \in \mathrm{GL}_k(R^0)$. Define the (extrinsic) Maurer-Cartan form

$$\mu_X = X^{-1}dX \in \mathrm{Mat}_{k \times k}(R^1) \quad (3.1)$$

and, for any $n \in \mathbb{Z}_{\geq 0}$, the Maurer-Cartan trace

$$\beta_X^n = \mathrm{tr}(\mu_X^n) \in R^n. \quad (3.2)$$

Consider the Maurer-Cartan equation, which may algebraically be written in the variable Y as

$$dY + Y^2 = 0. \quad (3.3)$$

Then the following properties hold.

Lemma 3.2. *The Maurer-Cartan form μ_X is a solution of the Maurer-Cartan equation (3.3).*

Proof. Since $XX^{-1} = I_k$, differentiation gives $X dX^{-1} + dXX^{-1} = 0$. Reordering to solve for dX^{-1} , we find that $dX^{-1} = -X^{-1}dXX^{-1}$. Plugging into def. 3.1 of the Maurer-Cartan form, we find

$$\begin{aligned} d\mu_X &= d(X^{-1}dX) = dX^{-1}dX - X^{-1}d^2X \\ &= -X^{-1}dXX^{-1}dX - X^{-1}d^2X \\ &= -\mu_X^2 - X^{-1}d^2X. \end{aligned}$$

By noting that our differential has degree 1, the ultimate term d^2X vanishes by

$$d^2X = d(I_k dX) = dI_k dX - I_k d^2X = -d^2X,$$

and we recover the Maurer-Cartan equation (3.3). $\square$

Corollary 3.3. *Even powers of the Maurer-Cartan form are closed under the differential, and odd powers satisfy the relation $d\mu_X^{2n-1} = -\mu_X^{2n}$ for any $n \in \mathbb{Z}_{\geq 1}$.*

Proof. The first claim follows by induction on $n \in \mathbb{Z}_{\geq 0}$. The base case follows trivially since $\mu_X^0 = I_k$ is the identity, whereby it is closed under the differential. Then, we compute

$$\begin{aligned} d\mu_X^{2n} &= d\mu_X \mu_X^{2n-1} - \mu_X d\mu_X^{2n-1} \\ &= d\mu_X \mu_X^{2n-1} - \mu_X d\mu_X \mu_X^{2n-2} + \mu_X^2 d\mu_X^{2n-2} \\ &\stackrel{\text{lemma 3.2}}{=} -\mu_X^{2n+1} + \mu_X^{2n+1} + \mu_X^2 d\mu_X^{2n-2}, \end{aligned}$$

whence the claim follows by induction for $d\mu_X^{2n-2}$. The case for odd powers now follows by simple computation

$$d(\mu_X d\mu_X^{2n-2}) = d\mu_X \mu_X^{2n-2} \stackrel{\text{lemma 3.2}}{=} -\mu_X^{2n}.$$

□

Maurer-Cartan traces satisfy a number of computationally useful properties.

Lemma 3.4. *The following hold for all $n \in \mathbb{Z}_{\geq 1}$.*

1. $\beta_X^n = \text{tr}(dXX^{-1})^n$,
2. $\beta_{X^{-1}}^n = (-1)^n \beta_X^n$,
3. $\beta_{X^T}^n = (-1)^{n(n-1)/2} \beta_X^n$,
4. $\beta_X^{2n} = 0$,
5. $d\beta_X^{2n+1} = 0$,
6. $\beta_{X_1 \oplus X_2}^n = \beta_{X_1}^n + \beta_{X_2}^n$.

Proof. The first property follows directly from def. 3.1 and cyclicity of the trace. The second follows from the fact that $\mu_{X^{-1}} = -dXX^{-1}$, as shown in the proof of lemma 3.2. To prove the third property, note

$$\beta_{X^T}^n = \text{tr}(dX^T X^{-T})^n = \text{tr}(\mu_X^T)^n$$

and

$$(\mu_X^n)^T = (-1)^{n(n-1)/2} (\mu_X^T)^n,$$

since transposition is an anti-homomorphism. The fourth property follows from cyclicity of the trace, from which the fifth follows by cor. 3.3. The last property follows directly from def. 3.1. □

Lemma 3.5. *Let $n \in \mathbb{Z}_{\geq 1}$ and $A \in \mathrm{GL}_k(R^0)$ be constant, i.e. $dA = 0$. Then the map $X \mapsto \beta_X^n$ is invariant under both left and right multiplication by A .*

Proof. Consider constant $A, B \in \mathrm{GL}_k(R^0)$. Then

$$\mu_{AXB}^n = ((AXB)^{-1} A dXB)^n = (B^{-1} X^{-1} dXB)^n = B^{-1} \mu_X^n B,$$

whence the claim follows by cyclicity of the trace. $\square$

The Maurer-Cartan traces satisfy the following scale invariance property, which may be understood as a projective invariance.

Lemma 3.6. *Let $\lambda \in (R^0)^\times$. Then, for all $n \in \mathbb{Z}_{\geq 1}$ and $X \in \mathrm{GL}_k(R^0)$, we have $\beta_{\lambda X}^{2n+1} = \beta_X^{2n+1}$, whereas for the special case of $n = 0$ it holds that $\beta_{\lambda X}^1 = \beta_X^1 + k\lambda^{-1} d\lambda$.*

Proof. Note first that

$$\mu_{\lambda X} = \mu_X + \lambda^{-1} d\lambda I_k,$$

from which the case $n = 0$ follows immediately by taking the trace. Turning to $n > 0$: Since $\deg(d\lambda) = 1$, it follows by graded-commutativity that $(d\lambda)^2 = 0$. But then it follows that $\mu_{\lambda X}^{2n} = \mu_X^{2n}$. Then, applying the above formula once more gives

$$\mu_{\lambda X}^{2n+1} = \mu_X^{2n+1} + \lambda^{-1} d\lambda \mu_X^{2n}.$$

The claim then follows by taking the trace and applying lemma 3.4. $\square$

There is a well-known vanishing result, recalled by [Bro21, prop. 4.5] from [Car07], which we recount below. The lemma proves useful when studying the vanishing of canonical scattering amplitudes.

Lemma 3.7. *Let $X \in \mathrm{GL}_k(R^0)$. Then $\beta_X^m = 0$ for all $m \geq 2k$.*

Furthermore, the following theorem is taken from [Bro21, thm. 5.2] without proof, since its derivation requires tools whose lengthy study falls beyond the scope of this thesis. It will prove useful later when we study the order of poles of a certain class of canonical differential forms.

Theorem 3.8 ([Bro21]). *Let $X \in \mathrm{GL}_k(R^0)$. Then*

$$\beta_X^1 = d \log(\det X)$$

and

$$\beta_X^{2n+1} \in \frac{1}{\det(X)^{n+1}} \Omega_{\mathbb{Q}[a_{i,j}]/\mathbb{Q}}^{2n+1}.$$

Supposing further that X is symmetric, we find

$$\beta_X^{4n+1} \in \frac{1}{\det(X)^{n+1}} \Omega_{\mathbb{Q}[a_{i,j}]/\mathbb{Q}}^{4n+1}.$$

3.2. Maurer-Cartan cohomology. Since the differential is of degree 1, it is nilpotent in second order, as shown during the proof of lemma 3.2. Therefore, we get a well-defined Chevalley-Eilenberg complex (R, d) , giving us a well-defined cohomology theory

$$H^\bullet(R) = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} H^n(R), \quad (3.4)$$

where the grading is inherited from the degree of elements in $R = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} R_n$. By lemma 3.2, the Maurer-Cartan traces β_X^{2n+1} are closed in this cohomology, giving potentially non-vanishing cohomology classes

$$[\beta_X^{2n+1}] \in H^{2n+1}(R). \quad (3.5)$$

In the sequel we will consider the case where $X = X^T$ is symmetric, which reduces the number of candidate classes. By lemma 3.4, we find for symmetric X that

$$\beta_X^{4n+3} = (-1)^{(4n+3)(4n+2)/2} \beta_X^{4n+3} = (-1)^{(4n+3)(2n+1)} \beta_X^{4n+3} = -\beta_X^{4n+3}, \quad (3.6)$$

whence, for all $n \in \mathbb{Z}_{\geq 0}$, the Maurer-Cartan traces β_X^{4n+3} are exact and vanish in cohomology. The remaining candidates for potentially non-vanishing elements in cohomology are

$$[\beta_X^{4n+1}] \in H^{4n+1}(R). \quad (3.7)$$

Recall that by lemma 3.6, only the classes β_X^n for $n > 1$ are projectively invariant. It will prove useful to require projective invariance of these classes, further restricting our candidate classes from (3.7) by disregarding β_X^1 .

3.3. Maurer-Cartan-Hopf algebra. Consider the map

$$\mathrm{GL}_n(R^0) \times \mathrm{GL}_m(R^0) \rightarrow \mathrm{GL}_{n+m}(R^0); (X_1, X_2) \mapsto X_1 \oplus X_2, \quad (3.8)$$

and, as $n, m \rightarrow \infty$, its direct limit counterpart

$$\mathrm{GL}_\infty(R^0) \times \mathrm{GL}_\infty(R^0) \rightarrow \mathrm{GL}_\infty(R^0); (X_1, X_2) \mapsto X_1 \oplus X_2. \quad (3.9)$$

Denoting by R_{MC} the differential algebra generated by Maurer-Cartan traces as given by def. 3.1, the map (3.9) gives rise to a coproduct

$$\Delta : R_{\mathrm{MC}} \rightarrow R_{\mathrm{MC}} \otimes R_{\mathrm{MC}}, \quad (3.10)$$

where, by lemma 3.4, primitive elements are given by

$$\Delta \beta^{2n+1} = \beta^{2n+1} \otimes 1 + 1 \otimes \beta^{2n+1}, \quad (3.11)$$

since, when acting on $X_1, X_2 \in \mathrm{GL}_\infty(R^0)$, the above statement follows from lemma 3.4, namely

$$\beta_{X_1 \oplus X_2}^{2n+1} = \beta_{X_1}^{2n+1} + \beta_{X_2}^{2n+1}.$$

4 Scattering amplitudes

4.1. Graph differential forms. After setting foundations in the previous sections, we are in a position to combine these results and define a class of differential forms on the graph complex, as pioneered by [Bro21]. These canonical differential forms allow us to define scattering amplitudes, analogous to those used in quantum field theory, over the graph complex.

Definition 4.1. Define the coefficient ring of the graph Laplacian

$$R_G^0 = \mathbb{Q}[(x_e)_{e \in E_G}, \Psi_G^{-1}], \quad (4.1)$$

where Ψ_G is the Kirchhoff polynomial from def. 2.5. Then, define the $\mathbb{Q}$ -algebra of Kähler differentials

$$R_G^\bullet = \Omega_{\mathbb{A}^{E_G} \setminus Z(\Psi_G)}^\bullet(\text{Spec}(R_G^0)), \quad (4.2)$$

where $\text{Spec}(R_G^0)$ denotes the spectrum of the ring R_G^0 and $\mathbb{A}^{E_G} \setminus Z(\Psi_G)$ the affine hypersurface complement of the zero locus $Z(\Psi_G)$ of the Kirchhoff polynomial.

Recall that the ring of Kähler differentials is a graded-commutative and graded-differential $\mathbb{Q}$ -algebra with degree 1 differential. Therefore, the assumptions of section 3 are fulfilled and we may define Maurer-Cartan forms over $R_G^\bullet$.

Definition 4.2. Let $\Lambda_G \in \text{GL}_{h_G}(R^0)$ be the graph Laplacian as in def. 2.4. Let $k \in \mathbb{Z}_{\geq 1}$. Define the Brown differential forms

$$\omega_G^{4k+1} = \beta_{\Lambda_G}^{4k+1} \in R_G^{4k+1}, \quad (4.3)$$

where $\beta_{\Lambda_G}^{4k+1}$ denote the Maurer-Cartan traces over $R_G^\bullet$ as in def. 3.1.

Recall that by equation (3.7) the only candidates for non-vanishing elements in cohomology are β_X^{4k+1} whenever $X = X^T$ symmetric. Since the graph Laplacian Λ_G is, by definition, symmetric, we may restrict to the Brown differential forms of degrees $(4k+1)$, as given in def. 4.2.

Lemma 4.3. *The Brown differential forms satisfy*

$$\omega_G^{4k+1} = \frac{N_G}{\Psi_G^{4k+1}} \quad (4.4)$$

for some $N_G \in \Omega^{4k+1}(\mathbb{Q}[(x_e)_{e \in E_G}])$.

Proof. Recall that, by def. 4.2, $\omega_G^{4k+1} = \text{tr}(\mu_{\Lambda_G}^{4k+1}) = \text{tr}((\Lambda_G^{-1} d\Lambda_G)^{4k+1})$. Recall also from linear algebra that for an invertible matrix A , the following relation between its inverse and adjoint holds

$$A^{-1} = \det(A)^{-1} \text{adj}(A).$$

Since Λ_G is, by construction, invertible, and $\det \Lambda_G = \Psi_G$, we find

$$\omega_G^{4k+1} = \det(\Lambda_G)^{-(4k+1)} \text{tr}((\text{adj}(\Lambda_G) d\Lambda_G)^{4k+1}) = \frac{\text{tr}((\text{adj}(\Lambda_G) d\Lambda_G)^{4k+1})}{\Psi_G^{4k+1}}.$$

Since the adjoint is the transpose of the cofactor matrix, and since Λ_G takes values in $\mathbb{Q}[(x_e)_{e \in E_G}]$, $\text{adj}(\Lambda_G)$ must also take values in $\mathbb{Q}[(x_e)_{e \in E_G}]$. Since $d\Lambda_G$ takes values in $\Omega^1(\mathbb{Q}[(x_e)_{e \in E_G}])$, we conclude that

$$N_G := \text{tr}((\text{adj}(\Lambda_G) d\Lambda_G)^{4k+1}) \in \Omega^{4k+1}(\mathbb{Q}[(x_e)_{e \in E_G}]),$$

and the claim follows. $\square$

Theorem 4.4 ([Bro21]). *The Brown differential forms ω_G^{4k+1} are well-defined. They are projectively invariant and factor through to closed differential forms*

$$\omega_G^{4k+1} \in \Omega_{\mathbb{P}^{E_G-1}(\mathbb{R}) \setminus Z(\Psi_G)}^{4k+1}(\text{Spec}(R_G^0)) \quad (4.5)$$

with singularities given by poles of order at most $k+1$ along $Z(\Psi_G)$.

Proof. Projective invariance for the Brown differential forms follows directly from the projective invariance of Maurer-Cartan traces by lemma 3.6. Factorisation to Kähler differentials over the projective space complement follows. Note that by lemma 3.4, $d\omega_G^{2n+1} = 0$, whence by the choice of $n = 2k$ it follows that the ω_G^{4k+1} are closed. By lemma 3.5, ω_G^{4k+1} are invariant under $\text{GL}_k(R^0)$ -actions, whence they are independent of the choice of basis for Λ_G , and thereby well-defined. By lemma 4.3, the singularities of ω_G^{4k+1} correspond to the zero locus $Z(\Psi_G)$. Since $\Psi_G \in \mathbb{Q}[(x_e)_{e \in E_G}]$, these singularities are poles. By thm. 3.8, they must be of order at most $(k+1)$. $\square$

Denote by $\sigma_G = \{(x_e)_{e \in E_G} : x_e > 0\} \subset \mathbb{P}^{E_G}(\mathbb{R})$ the open coordinate simplex. By def. 2.5, the Kirchhoff polynomial is a non-trivial sum of positive monomial terms. Therefore, $\sigma_G \cap Z(\Psi_G) = \emptyset$. Combined with thm. 4.4, we get the following corollary.

Corollary 4.5. *The Brown differential forms ω_G^{4k+1} are smooth on the open coordinate simplex σ_G .*

4.2. Graph-theoretic identities. There are many identities that highlight the compatibility of Brown differential forms between different actions on the graph complex. The following highlights the invariance of Brown differential forms under graph automorphisms.

Lemma 4.6. *Let $\pi \in \text{Aut}(G)$. Then the induced isomorphism $\pi^* : R_G^0 \rightarrow R_G^0$, with $\pi^* x_e = x_{\pi(e)}$, satisfies*

$$\omega_G^{4n+1} = \pi^* \omega_G^{4n+1}.$$

Proof. The automorphism π induces an automorphism P of the graph homology $H_1(G; \mathbb{Z})$, whereby it acts on the graph Laplacian via $\pi^* \Lambda_G = P^T \Lambda_G P$. By lemma 3.5, the claim follows. $\square$

Furthermore, Brown differential forms behave nicely under restriction to linear subspaces.

Proposition 4.7. *Let $\gamma \subset E_G$ with $h_\gamma = 0$. Then, as meromorphic forms on the linear subspace $L_\gamma = \mathbb{P}^{E_G/\gamma} \subset \mathbb{P}^{E_G}$, it holds that*

$$\omega_G^{4n+1} \Big|_{L_\gamma} = \omega_{G/\gamma}^{4n+1}.$$

Proof. We may assume that γ consists of only a single edge e , since by regularity of ω^{4n+1} at a generic point of L_γ , the statement may be reduced to iteratively contracting edge-wise for all edges in γ . The statement thus reduces to showing that

$$\omega_G^{4n+1} \Big|_{x_e=0} = \omega_{G/e}^{4n+1}.$$

Since e has distinct endpoints, its contraction defines an isomorphism $H_1(G; \mathbb{Z}) \cong H_1(G/e; \mathbb{Z})$, whence by def. 2.4 of the graph Laplacian, we find

$$\Lambda_G \Big|_{x_e=0} = \Lambda_{G/e}.$$

But by def. 4.2 of ω_G , the statement thus follows. $\square$

4.3. Algebraic differential forms. The above compatibility properties allow us to define families of Brown differential forms that naturally encode this desirable behaviour. To this end, we first define a blow-up space to deal with singularities of the Brown differential forms.

Definition 4.8. Define the total blow-up space

$$P^{\text{tot}} = (P^G)_{G \in \mathbb{GC}_2} \quad (4.6)$$

to be the collection of schemes P^G of iterated blow-ups

$$\pi_G : P^G \longrightarrow \mathbb{P}^{E_G} \quad (4.7)$$

along linear subspaces L_γ in increasing order of dimension for all graphs $G \in \mathbb{GC}_2$ in the graph complex together with the inclusion morphisms

$$\begin{aligned} \iota_e : P^{G/e} &\hookrightarrow P^G, \\ \iota_\gamma : P^\gamma \times P^{G/\gamma} &\hookrightarrow P^G. \end{aligned} \quad (4.8)$$

Note that in reality we consider only iterated blow-ups along linear subspaces corresponding to $\gamma \in \mathbb{B}_G^{\text{core}}$, where $\mathbb{B}_G^{\text{core}} \subset 2^{E_G}$ is a certain edge subset. However, for the purposes of the present thesis, these details are only of secondary import and we refer the interested reader to [Bro17; Bro21]. On these total blow-up spaces, we now define suitable families of Brown differential forms.

Definition 4.9. Define a primitive algebraic k -form $(\omega_G)_{G \in \mathbb{GC}_2}$ on P^{tot} to be a collection of Brown differential forms such that

1. for all $G \in \mathbb{GC}_2$, ω_G is meromorphic on P^G and smooth when restricted to the open coordinate simplex σ_G , and
2. the restriction of ω_G along the inclusion morphisms (4.8) satisfies

$$\begin{aligned} \iota_e^* \omega_G &= \omega_{G/e}, \\ \iota_\gamma^* \omega_G &= \omega_\gamma \wedge 1 + 1 \wedge \omega_{G/\gamma}. \end{aligned}$$

Furthermore, we require that for all $\tau \in \text{Aut}(G)$ it holds that $\tau^* \omega_G = \omega_G$. A general algebraic k -form on P^{tot} is, then, defined to be an exterior product of primitive algebraic k -forms.

4.4. Brown-Hopf algebra. Consider the graded exterior $\mathbb{Z}$ -algebra

$$\Omega_{\text{alg}}^\bullet = \bigoplus_{d \geq 0} \Omega_{\text{alg}}^d, \quad (4.9)$$

generated by symbols β^{4n+1} for $n \geq 1$. On $\Omega_{\text{alg}}^\bullet$, we introduce a coproduct structure

$$\Delta : \Omega_{\text{alg}}^\bullet \longrightarrow \Omega_{\text{alg}}^\bullet \otimes_{\mathbb{Z}} \Omega_{\text{alg}}^\bullet, \quad (4.10)$$

defined by the requirement that the generators be primitive

$$\Delta\beta^{4n+1} = \beta^{4n+1} \otimes 1 + 1 \otimes \beta^{4n+1}. \quad (4.11)$$

Note that we may think of the exterior algebra of algebraic k -forms, as per def. 4.9, in terms of the algebra $\Omega_{\text{alg}}^\bullet$ in equation (4.9). Using this identification, it is recognised that the coproduct structure (4.10) on algebraic forms is the analogue to that of section 3.3 on the differential algebra R_{MC} of Maurer-Cartan traces. We frequently make use of Sweedler notation

$$\Delta\omega = \sum_j \omega'_j \otimes \omega''_j \quad (4.12)$$

for algebraic forms $\omega \in \Omega_{\text{alg}}^\bullet$.

4.5. Canonical integrals. In the following, we take a closed algebraic k -form $(\omega_G)_{G \in \mathbb{GC}_2}$, as per def. 4.9, as given.

Definition 4.10. Let (G, η) be an oriented graph with $k+1$ edges. We define the canonical scattering amplitude

$$I_{(G, \eta)}((\omega_G)_{G \in \mathbb{GC}_2}) = \int_{\sigma_G} \omega_G$$

with the orientation on ω_G induced by η .

Since, in def. 4.9, we require that algebraic k -forms be smooth on ω_G , and since the domain ω_G is compact, canonical scattering amplitudes, as per def. 4.10, are by construction always finite. Note that this is in stark contrast to Feynman amplitudes in quantum field theory, where it is very much the norm that scattering amplitudes diverge and require a treatment by renormalisation methods. Moreover, canonical scattering amplitudes are not only finite by construction, they lift to well-defined objects on the graph complex.

Lemma 4.11. *Canonical scattering amplitudes $I_{(G, \eta)}$ are well-defined on the level of the graph complex $\mathbb{GC}_2$, i.e. they lift to functions $I_{[G, \eta]}$ on equivalence classes.*

Proof. Note first that reversal of orientation changes the canonical scattering amplitude by a sign, in line with def. 2.1. Now, let $\tau \in \text{Aut}(G)$. By lemma 4.6, we get the functoriality property

$$\tau^*(\omega_G)_{G \in \mathbb{GC}_2} = (\omega_G)_{G \in \mathbb{GC}_2},$$

whereby

$$I_{(G, \eta)}((\omega_G)_{G \in \mathbb{GC}_2}) = I_{(G, \tau(\eta))}((\omega_G)_{G \in \mathbb{GC}_2}),$$

and the claim follows. $\square$

Lemma 4.12. *Suppose ω is primitive. Then, the canonical scattering amplitude vanishes, unless $\deg(G) = 0$.*

Proof. By lemma 3.7, $\omega_G = 0$ unless $\deg \omega_G < 2h_G$. In order that the integral be well-defined, it must hold that $\deg \omega_G = e_G - 1$, whence $e_G - 2h_G \leq 0$. To show the remaining inequality, note that Λ_G may be represented by a matrix L_G of size $v_G - 1$ via the dual Laplacian construction. For details, we refer to [Bro21, (3.4)]. Therefore, using again lemma 3.7, we find that $\omega_G = 0$ unless $\deg \omega_G < 2(v_G - 1)$. The claim follows by noting the graph-theoretic identity $v_G = e_G - h_G + 1$, which implies that $e_G - 2h_G \geq 0$. $\square$

As a consequence, primitive forms may detect elements in zeroth homology only, whereas non-primitive forms may detect elements in higher homology. Computations may, in many cases, be simplified by considering the following version of Stokes' theorem for canonical scattering amplitudes.

Theorem 4.13. *Suppose G has $k + 2$ edges. Then*

$$\sum_{e \in E_G} \int_{\sigma_{G/e}} \omega_{G/e} + \sum_j \left(\sum_{\substack{\gamma \subset G \\ \text{core}}} \int_{\sigma_\gamma} (\omega'_j)_\gamma \times \int_{\sigma_{G/\gamma}} (\omega''_j)_{G/\gamma} \right) = 0,$$

where the second sum ranges over all core subgraphs $\gamma \subset G$ with $e_\gamma = \deg \omega'_j + 1$ and the orientations on the simplices are induced by that of G .

Proof. From Stokes' theorem it follows that

$$0 = \int_{\sigma_G} d\omega = \int_{\partial\sigma_G} \omega.$$

Now, $\partial\sigma_G$ may be written as a union of $\sigma_{G/e}$ for $e \in E_G$ and $\sigma_\gamma \times \sigma_{G/\gamma}$ for core subgraphs $\gamma \subset G$. Since

$$\omega \Big|_{\sigma_\gamma \times \sigma_{G/\gamma}} = \sum_j \omega'_j \Big|_{\sigma_\gamma} \wedge \omega''_j \Big|_{\sigma_{G/\gamma}},$$

in Sweedler notation, the theorem follows. $\square$

Corollary 4.14. *Suppose G has no tadpoles. Then, thm. 4.13 may be rewritten as*

$$\sum_{e \in E_G} \left(\int_{\sigma_{G/e}} \omega_{G/e} + \int_{\sigma_{G \setminus e}} \omega_{G \setminus e} \right) + \sum_{\gamma \subset G} \int_{\sigma_\gamma \times \sigma_{G/\gamma}} \Delta^{\text{red}} \omega = 0,$$

where $\Delta^{\text{red}} \omega = \Delta \omega - 1 \otimes \omega - \omega \otimes 1$ denotes the reduced coproduct.

Non-zero homology classes may now be detected in a comparatively simple manner, as highlighted in the following corollary.

Corollary 4.15. *Let G be of degree 0, closed, and with edge grading e , and let ω be a primitive algebraic form of degree $e - 1$. Suppose*

$$I_G(\omega) = \int_{\sigma_G} \omega_G \neq 0.$$

Then, the homology class $[G] \in H_0(\mathbb{GC}_2)$ is non-zero.

Using diagram chasing methods, similar results to cor. 4.15 exist also in higher homological degrees.

5 Some known results on homology

While there are numerous mysteries regarding the nature of graph homology, many results are already known throughout the literature. We give an overview of some important results regarding the graph homology, though without recounting the proofs, which would undoubtedly go well beyond the scope of this thesis. Firstly, it is known that the homology vanishes in negative degrees.

Proposition 5.1 ([Wil15]). *Let $n < 0$ and $h \geq 1$. Then, $H_n(\mathbb{GC}_2)^{(h)} = 0$.*

One of the simplest families of graphs to study is the family of wheels. It follows directly from def. 2.1 that the even wheel classes vanish, since they admit an odd automorphism. However, it turns out that the remaining odd wheel classes generate numerous non-vanishing elements in homology.

Proposition 5.2 ([KWŽ17]). *Denote by W_n the wheel with n spokes. Then, the odd wheel classes $[W_{2n+1}] \in H_0(\mathbb{GC}_2)$ are non-zero for all $n \in \mathbb{Z}_{>0}$.*

In fact, conjecturally one can say more about the canonical scattering amplitudes on the odd wheel classes.

Conjecture 5.3 ([BS21; Bro21]). *Let ω be an algebraic form. Denote by ζ the Riemann zeta function. Then, the canonical scattering amplitudes for the odd wheel classes read*

$$I_{W_{2n+1}}(\omega^{4n+1}) = (2n+1) \binom{4n+2}{2n+1} \zeta(2n+1).$$

Conjecture 5.3 has been confirmed numerically in [BS21] to some extent. The odd wheel classes give the odd-dimensional contributions to homology in zeroth dimension, as illustrated by table 1. The remaining known classes given by the table are presumably only representable as linear combinations of graphs. Not much more is known systematically above loop order $h_G \geq 11$, highlighting the complexity of graph homology as a mathematical object. In that spirit, the following conjecture illustrates the magnitude that graph homology is believed to exhibit.

Conjecture 5.4 ([Bro21]). *There exists a non-canonical embedding*

$$\text{FreeLie}(\Omega_{\text{alg}}^\bullet) \hookrightarrow H^\bullet(\mathbb{GC}_2) \tag{5.1}$$

of graded Lie algebras.

Conjecture 5.4 may be thought of as a generalisation of the classical conjecture by Deligne and Ihara, made into a theorem by Brown, which reads as follows.

$H_8(\mathbb{GC}_2)$										0
$H_7(\mathbb{GC}_2)$								0	$\boxed{1}$	
$H_6(\mathbb{GC}_2)$							0	0	0	
$H_5(\mathbb{GC}_2)$						0	0	0	0	
$H_4(\mathbb{GC}_2)$					0	0	0	0	0	
$H_3(\mathbb{GC}_2)$				0	$\boxed{1}$	0	$\boxed{1}$	$\boxed{1}$	$\boxed{2}$	
$H_2(\mathbb{GC}_2)$			0	0	0	0	0	0	0	
$H_1(\mathbb{GC}_2)$		0	0	0	0	0	0	0	0	
$H_0(\mathbb{GC}_2)$	0	$\textcircled{1}$	0	$\textcircled{1}$	0	$\textcircled{1}$	$\textcircled{1}$	$\textcircled{1}$	$\textcircled{1}$	
h_G	1	2	3	4	5	6	7	8	9	10

Table 1: [Bro21] Known dimensions of $H_n(\mathbb{GC}_2)$ at genus $h_G \leq 10$. Boxed results have only been validated numerically, whereas circled results have been computed analytically.

Theorem 5.5 ([Iha89; Bro12]). *There exists an embedding*

$$\text{FreeLie}((\sigma_{2n+1})_{n \in \mathbb{Z}_{>0}}) \cong \text{FreeLie}(\Omega_{\text{alg}}^{\text{prim}}) \hookrightarrow H^0(\mathbb{GC}_2) \quad (5.2)$$

of Lie algebras.

Furthermore, $H^0(\mathbb{GC}_2)$ is known to be related to the Grothendieck-Teichmüller Lie algebra grt , defined as the tangent Lie algebra to the Grothendieck-Teichmüller group as introduced by [Dri90]. The construction of the Grothendieck program goes well beyond the scope of this thesis and we refer the interested reader instead to [DR12; Fre17]. However, the result reads as follows.

Theorem 5.6 ([Wil15]).

$$H^0(\mathbb{GC}_2) \cong \text{grt}. \quad (5.3)$$

As such, we may think of the Brown-Deligne-Ihara thm. 5.5 as stating the existence of an embedding

$$\text{FreeLie}((\sigma_{2n+1})_{n \in \mathbb{Z}_{\geq 0}}) \hookrightarrow \text{grt}, \quad (5.4)$$

of which it is conjectured that it be an isomorphism of Lie algebras.

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